Chapter 11

Compressible Flow

Introduction

- Compressible flow –variable density, and equation of state is important
- Ideal gas equation of state—simple yet representative of actual gases at pressures and temperatures of interest
- Energy equation is important, due to the significant variation of temperature.

11.1 Ideal gas relationship

$$P = \rho RT$$

For ideal gas, **internal energy** *u*=*u*(*T*)

constant pressure specific heat: $c_\upsilon = (\frac{\partial u}{\partial T})_\upsilon = \frac{du}{dT}$

$$du = c_\upsilon dT \quad \rightarrow \quad u_2 - u_1 = \int_{T_1}^{T_2} c_\upsilon dT$$

-For moderate changes in temperature:

$$u_2 - u_1 = c_\upsilon (T_2 - T_1)$$

Enthalpy *h=h(T)*

$$h = u + \frac{p}{\rho} = u(T) + RT = h(T)$$

constant pressure specific heat: $c_p = (\frac{\partial h}{\partial T})_p = \frac{dh}{dT}$

$$dh = c_p dT \quad \rightarrow \quad h_2 - h_1 = \int_{T_1}^{T_2} c_p dT$$

-For moderate changes in temperature

$$h_2 - h_1 = c_p (T_2 - T_1)$$

Since *h=u+RT*, *dh=du+RdT*

or

$$\frac{dh}{dT} = \frac{du}{dT} + R \quad \rightarrow \quad c_p - c_\upsilon = R$$

$$k = \frac{c_p}{c_\upsilon} \ (\text{□ } 1.4 \text{ for air}) \quad \therefore c_p = \frac{Rk}{k-1} \ \text{ and } \ c_\upsilon = \frac{R}{k-1}$$

Entropy

1st *Tds* equation

$$Tds = du + pd(1/\rho)$$

$$\because h = u + \frac{p}{\rho} \quad \rightarrow \quad dh = du + pd(1/\rho) + \frac{1}{\rho}dp$$

$$\therefore Tds = dh - \frac{1}{\rho}dp$$ **---2nd Tds equation**

$$ds = \frac{du}{T} + \frac{p}{T}d(1/\rho) = c_\upsilon \frac{dT}{T} + \frac{R}{(1/\rho)}d(1/\rho)$$

$$= \frac{dh}{T} - \frac{(1/\rho)}{T}dp = \frac{c_p dT}{T} - \frac{R}{p}dp$$

For constant $c_p,\ c_v$:

$$s_2 - s_1 = c_\upsilon \ln\frac{T_2}{T_1} + R\ln(\frac{\rho_1}{\rho_2})$$

$$= c_p \ln\frac{T_2}{T_1} - R\ln(\frac{p_2}{p_1})$$

For adiabatic and frictionless flow of any fluid

$$ds = 0 \quad \text{or} \quad s_2 - s_1 = 0 \quad \leftarrow \quad \text{isentropic flow}$$

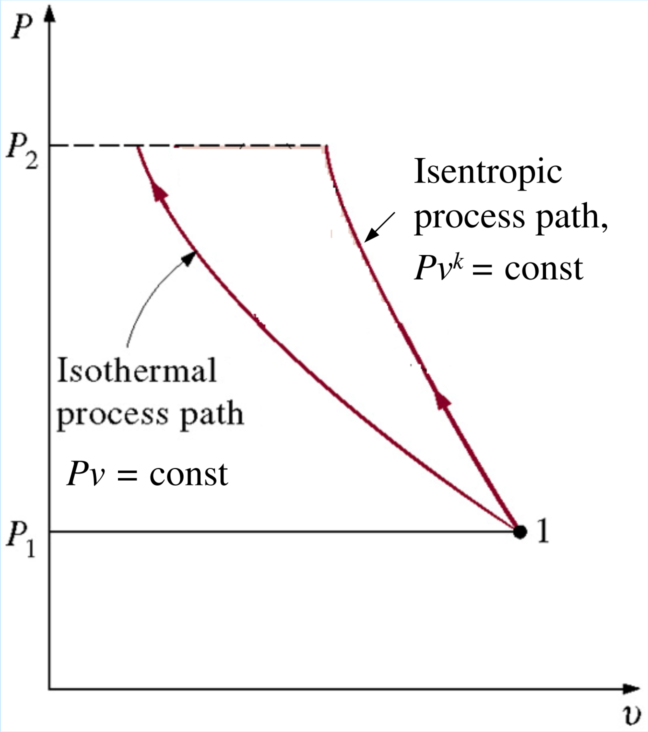
$$\text{or } c_\upsilon \ln\left(\frac{T_2}{T_1}\right) + R\ln\left(\frac{\rho_1}{\rho_2}\right) = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right) = 0$$

$$\frac{R}{k-1}\ln\left(\frac{T_2}{T_1}\right) = R\ln\left(\frac{\rho_2}{\rho_1}\right) \quad \rightarrow \quad \left(\frac{T_2}{T_1}\right)^{\frac{k}{k-1}} = \left(\frac{\rho_2}{\rho_1}\right)^k$$

$$\frac{kR}{k-1}\ln\left(\frac{T_2}{T_1}\right) = R\ln\left(\frac{p_2}{p_1}\right) \quad \rightarrow \quad \left(\frac{T_2}{T_1}\right)^{\frac{k}{k-1}} = \left(\frac{p_2}{p_1}\right)^k$$

$$\therefore \left(\frac{T_2}{T_1}\right)^{\frac{k}{k-1}} = \left(\frac{\rho_2}{\rho_1}\right)^k = \left(\frac{p_2}{p_1}\right) \Rightarrow \frac{p}{\rho^k} = \text{const}, \quad \text{for isentropic flow} \qquad (11.25)$$

Comparison of isentropic and isothermal compression

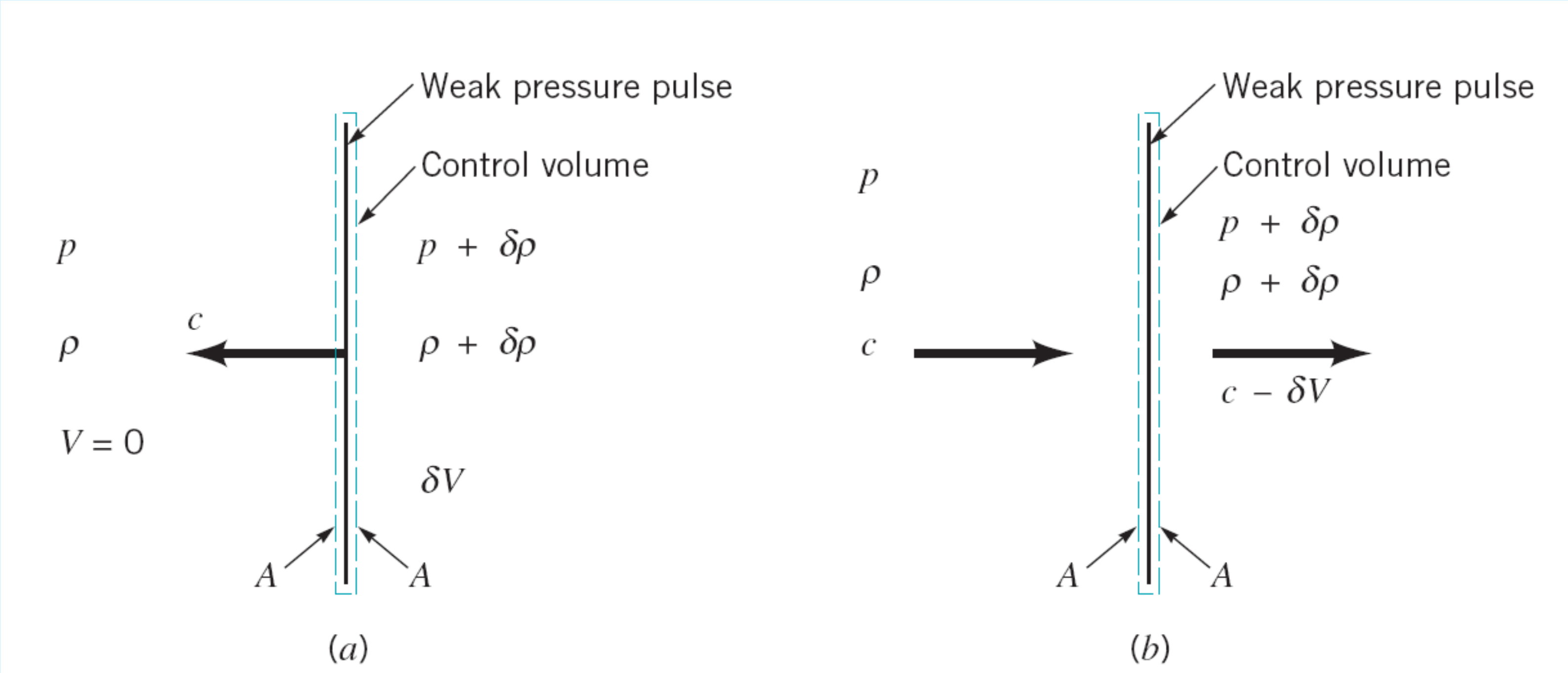


11.2 Mach number and speed of sound

$$\text{Ma} = \frac{V}{c}, \quad V\text{--local flow velocity,} \;\; c\text{--speed of sound}$$

Sound generally consists of weak pressure pulses that move through air.

Consider 1-D of infinitesimally thin weak pressure pulse moving at the speed of sound through a fluid at rest.



fluid at rest

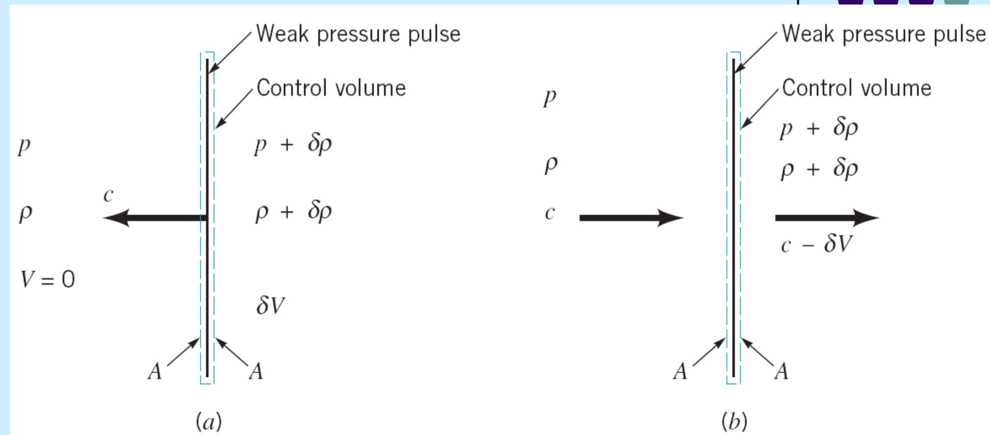
Observer moving with control volume

-conservation of mass

$$\rho A c = (\rho + \delta\rho) A (c - \delta V)$$

$$\rho c = \rho c - \rho\delta V + c\delta\rho - \cancel{\delta\rho\delta V}$$

$$\rho\delta V = c\delta\rho$$



-linear momentum conservation

$$-c\rho c A + (c - \delta V)(\rho + \delta\rho)(c - \delta V) A = pA - (p + \delta p) A$$

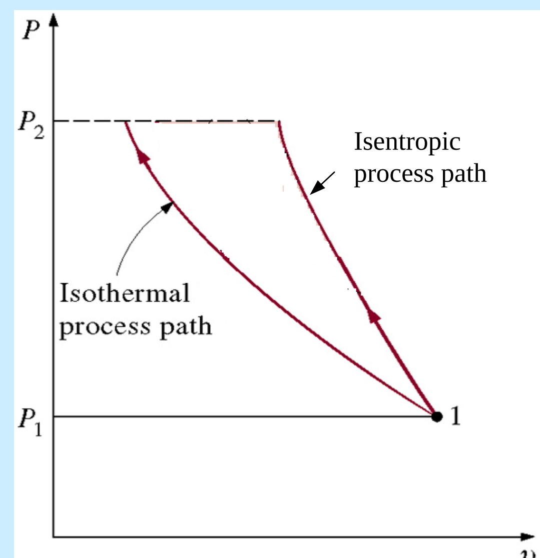
$$\because (\rho + \delta\rho)(c - \delta V) A = \rho c A \qquad \text{(continuity)}$$

$$\therefore -c\rho c A + (c - \delta V)\rho c A = -\delta p A$$

$$-\delta V \rho A c = -\delta p A \quad \rightarrow \quad \rho\delta V = \delta p / c$$

$$\because \rho\delta V = c\delta\rho \qquad \text{(continuity)}$$

$$\therefore c\delta\rho = \frac{\delta p}{c} \quad \rightarrow \quad c^2 = \frac{\delta p}{\delta\rho} \quad \Rightarrow \quad c = \sqrt{\frac{\delta p}{\delta\rho}}$$



-Alternative derivation using conservation of energy instead of momentum equation

$$\frac{\delta p}{\rho} + \delta\left(\frac{V^2}{2}\right) - g\delta z = \delta(\text{loss}) \quad \text{--from (5.103)}$$

$\delta(\text{loss}) \approx 0$ for frictionless flow; and $\delta(\text{z}) \approx 0$

$$\therefore \frac{\delta p}{\rho} + \frac{(c-\delta V)^2}{2} - \frac{c^2}{2} = 0 \quad \rightarrow \quad \rho\delta V = \frac{\delta p}{c}$$

$$\text{or} \quad \frac{\delta p}{\rho} = c\delta V$$

$$\rightarrow \frac{\delta p}{c} = \rho\delta V = c\delta\rho \quad (\text{from continuity: } \rho\delta V = c\delta\rho)$$

$$\therefore c^2 = \frac{\delta P}{\delta\rho} \Rightarrow c = \sqrt{\frac{\delta p}{\delta\rho}}$$

Assume the frictionless flow through the control volume is adiabatic, then the flow is isentropic.

In the limit $\delta p \to \partial p \to 0$

$$c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} \qquad (11.34)$$

- For isentropic flow of idea gas $p = c\rho^k$

$$\left(\frac{\partial p}{\partial \rho}\right)_s = ck\rho^{k-1} = \frac{p}{\rho^k}k\rho^{k-1} = \frac{p}{\rho}k = RTk \Rightarrow \quad c = \sqrt{RTk} \qquad (11.36)$$

- More generally, use bulk modulus of elasticity

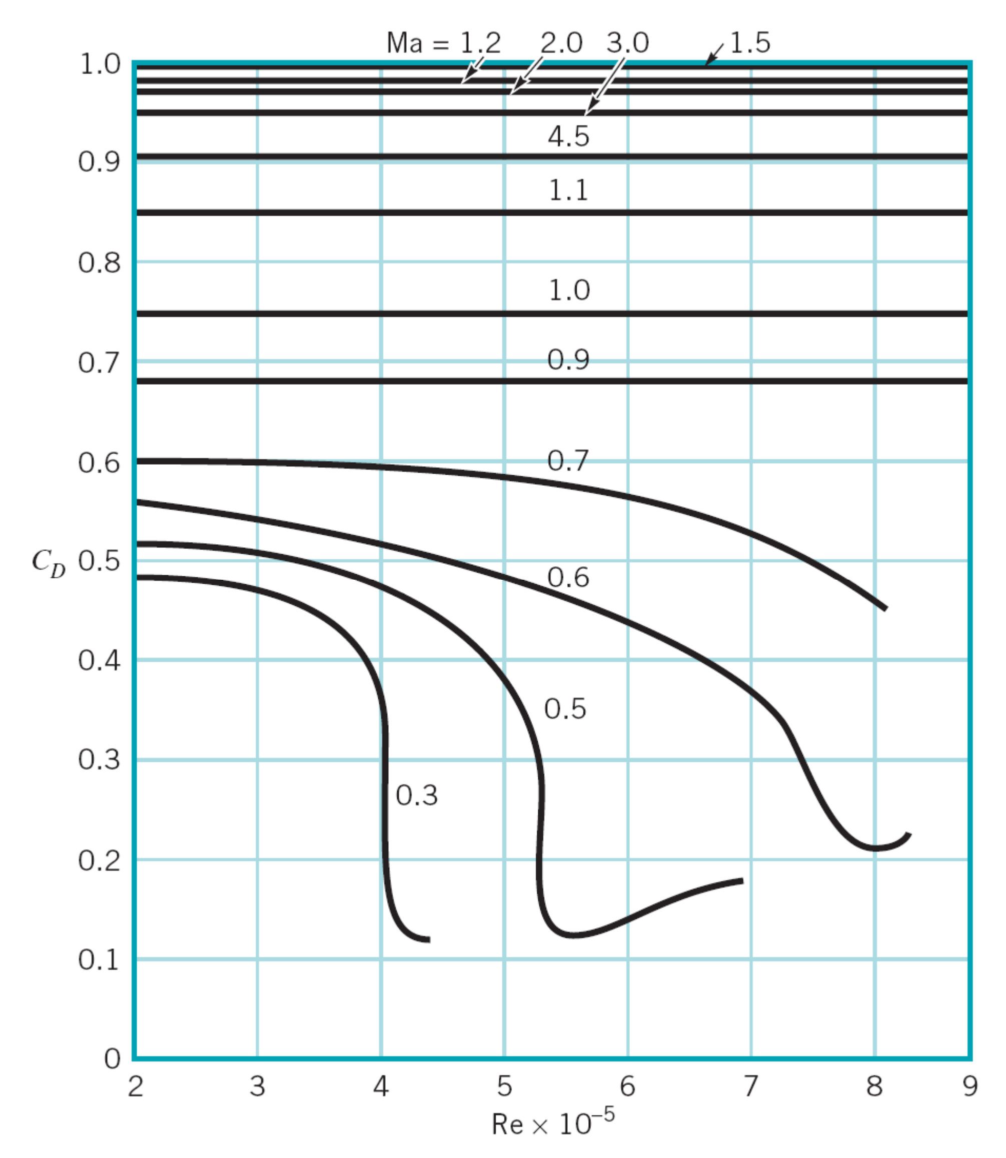
$$E_v = \frac{dp}{d\rho / \rho} = \rho\left(\frac{\partial p}{\partial \rho}\right)_s$$

$$\therefore c = \sqrt{E_v / \rho}$$

- For incompressible flow $E_v \to \infty \qquad \therefore c \to \infty$

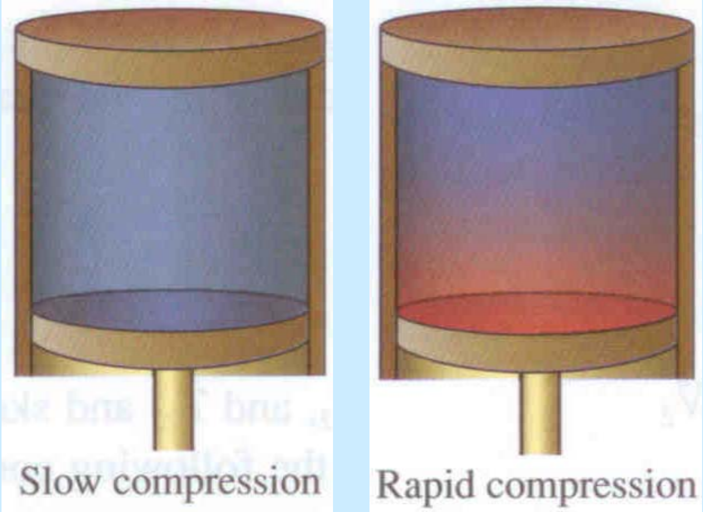
11.3 Category of compressible flow

- effect of compressibility on C_D of a sphere



Why C_D increases with Ma?

Can you explain physically?

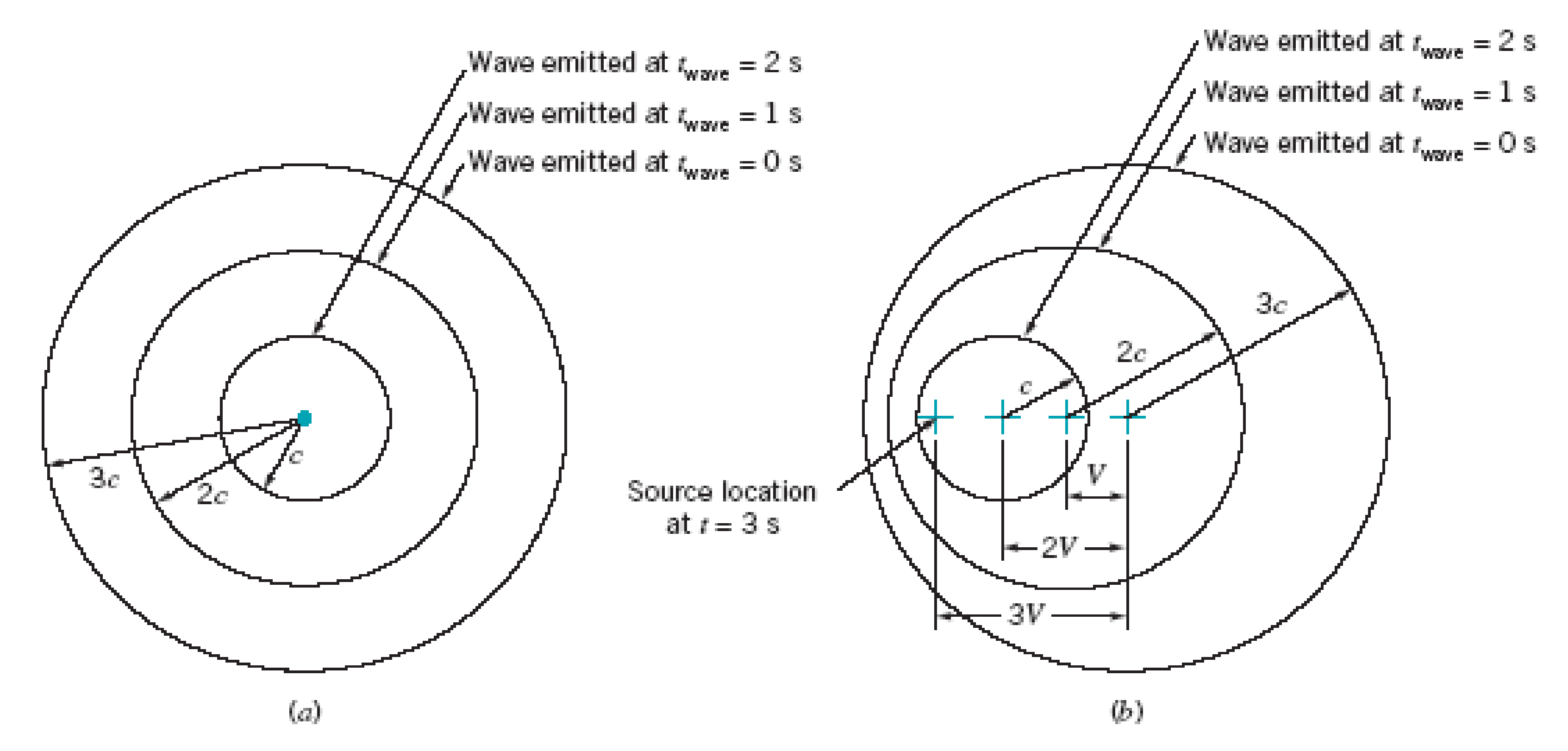


(from S.R. Turns, Thermal-Fluid Sciences)

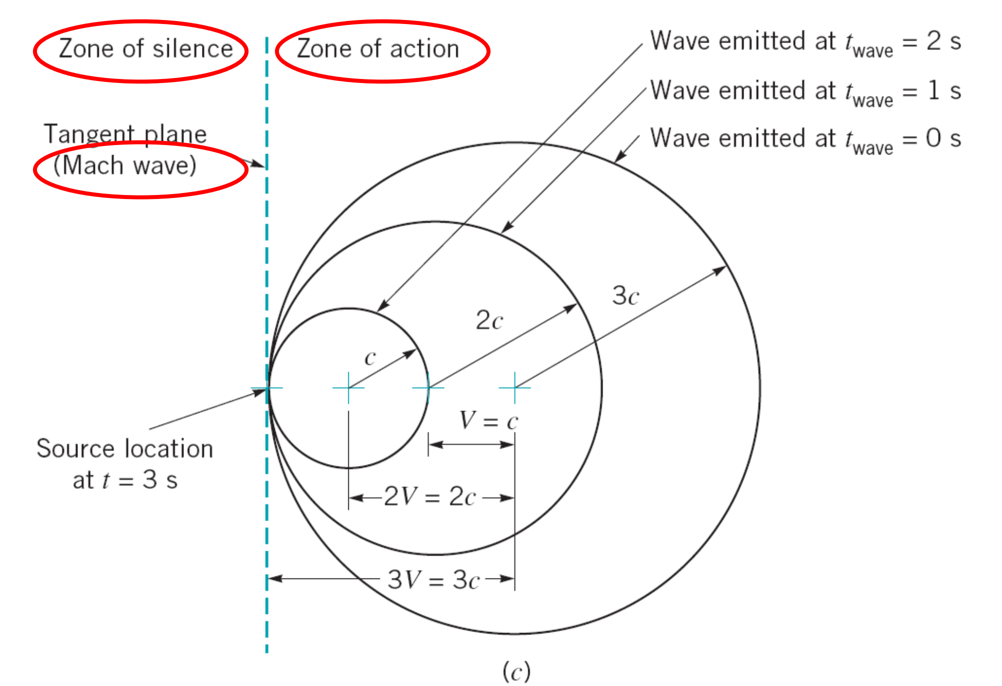
- Imagine the emission of weak pressure pulses from a point source

$$r = (t - t_{wave})c$$

where $t \rightarrow$ present time, $t_{wave} \rightarrow$ time wave emitted

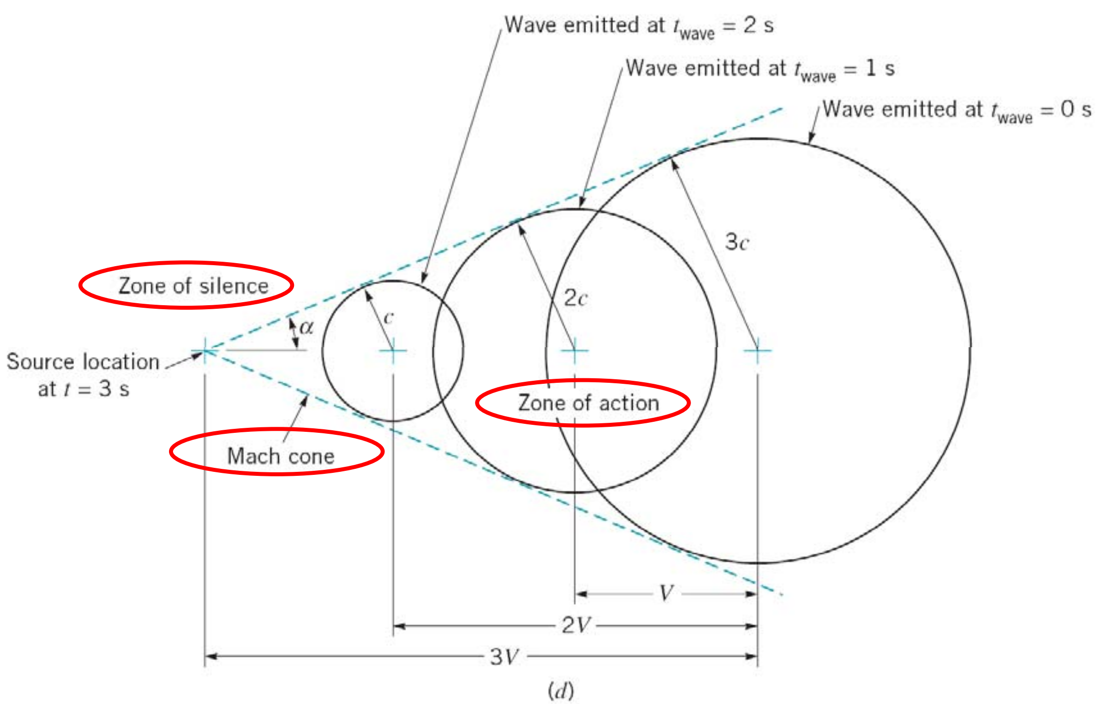


- stationary point source - moving point source V< C



(c)

- source moving at *V*=c



(d)

- source moving at *V*>c

- Category of fluid flow

1. Incompressible flow $\mathrm{Ma} \le 0.3$

   unrestricted, linear symmetrical and instantaneous pressure communication.

2. Compressible subsonic flow $0.3 < \mathrm{Ma} < 1.0$

   unrestricted, but noticeably asymmetrical pressure communication

3. Compressible supersonic flow $\mathrm{Ma} \ge 1$

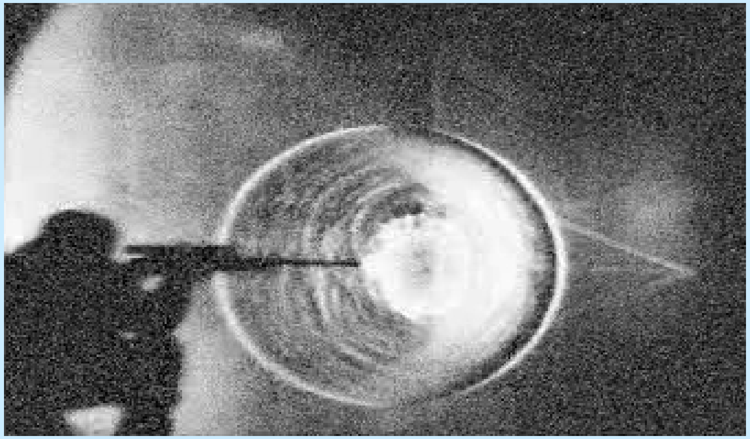
   formation of Mach wave, pressure communication restricted to zone of action

4. transonic flow $0.9 \le \mathrm{Ma} \le 1.2$ (modern aircraft)

5. hypersonic flow $\mathrm{Ma} \ge 5$ (space shuttle)

Example 11.4 Mach cone

Mach cone from a rifle bullet



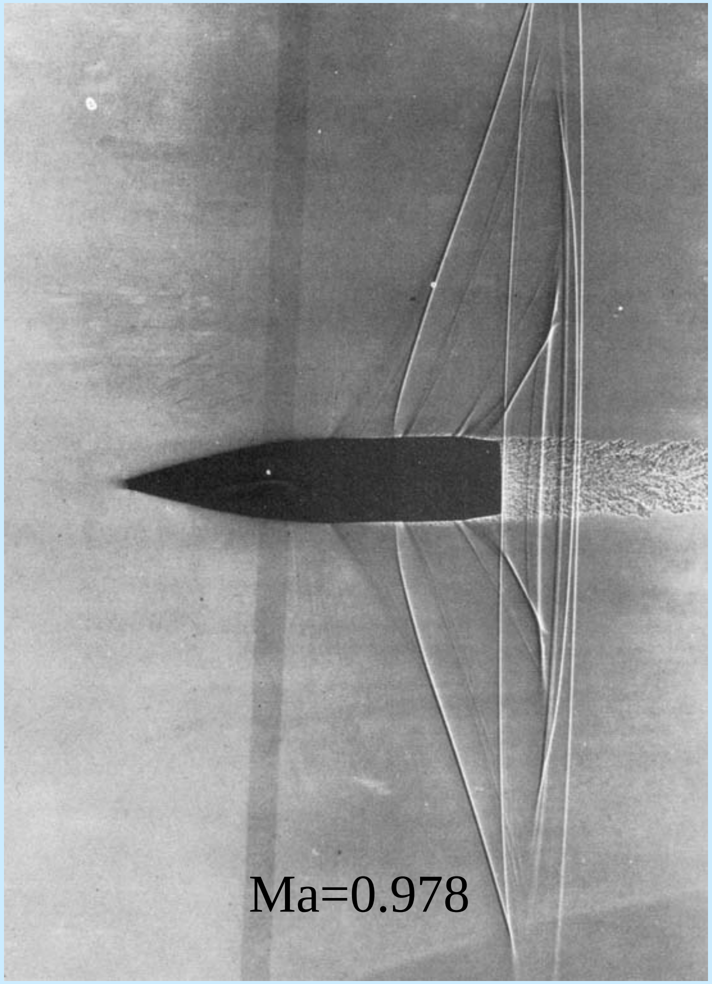
from Gas Dynamics Lab, The Penn. State University, 2004

from M. Van Dyke, An Album of Fluid Motion

Video: Mach cone of an airplane

(Note the condensed cloud across the shock wave.)

(Can you estimate the airplane speed?)



11.4 Isentropic flow of an idea gas - no heat transfer and frictionless

11.4.1 Effect of variation in flow cross-section area

- conservation of mass

$$\dot{m} = \rho A V = \text{const.}$$

- Conservation of momentum for a inviscid and steady flow

$$dp + \frac{1}{2}\rho d(V^2) + \gamma \cancel{dz}^{\,0} = 0$$

$$\frac{dp}{\rho V^2} = -\frac{dV}{V}$$

$$\text{Since } \dot{m} = \rho A V = c, \quad \therefore \ln \rho + \ln A + \ln V = c'$$

$$\text{differentiation} \rightarrow \frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0$$

$$\rightarrow \quad -\frac{dV}{V} = \frac{d\rho}{\rho} + \frac{dA}{A} \; \left(= \frac{dp}{\rho V^2}\right) \qquad (11.44)$$

$$\frac{dA}{A} = \frac{dp}{\rho V^2} - \frac{d\rho}{\rho} = \frac{dp}{\rho V^2}\left(1 - \frac{d\rho}{\rho}\cdot\frac{\rho V^2}{dp}\right)$$

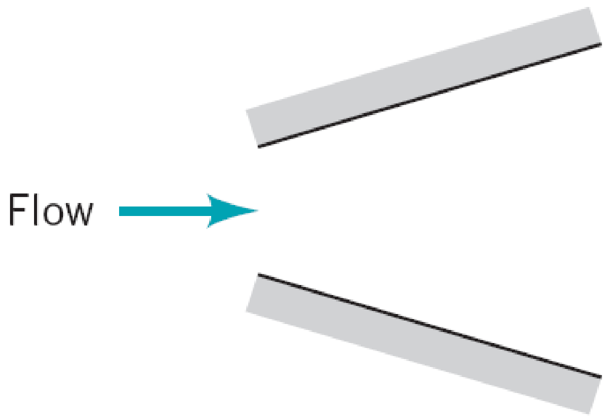
$$= \frac{dp}{\rho V^2}\left(1 - \frac{V^2}{dp/d\rho}\right) \qquad (11.45)$$

Since $c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}$, and $\text{Ma} = \frac{V}{c}$

$$\therefore \frac{dp}{\rho V^2}(1 - \text{Ma}^2) = \frac{dA}{A} \qquad (11.47)$$

$$-\frac{dp}{\rho V^2} = -\frac{dA}{A}\frac{1}{1 - \text{Ma}^2} = \frac{dV}{V} \qquad (11.48)$$

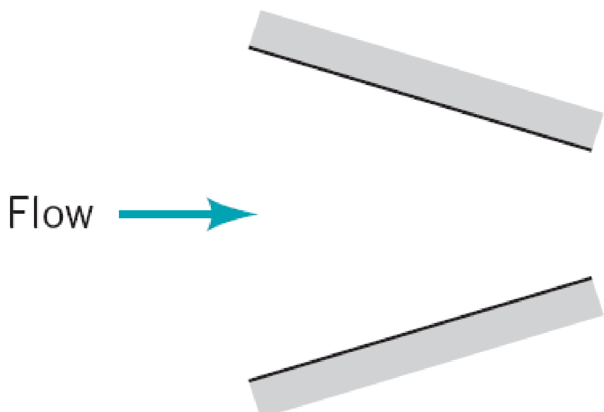
diverging duct



	Subsonic flow ($Ma < 1$)	Supersonic flow ($Ma > 1$)
(a)	$dA > 0$ $dV < 0$	$dA > 0$ $dV > 0$

$$-\frac{dp}{\rho V^2} = -\frac{dA}{A}\frac{1}{1-\mathrm{Ma}^2} = \frac{dV}{V}$$

converging duct



	Subsonic flow	Supersonic flow
(b)	$dA < 0$ $dV > 0$	$dA < 0$ $dV < 0$

Combining (11.44) and (11.48):

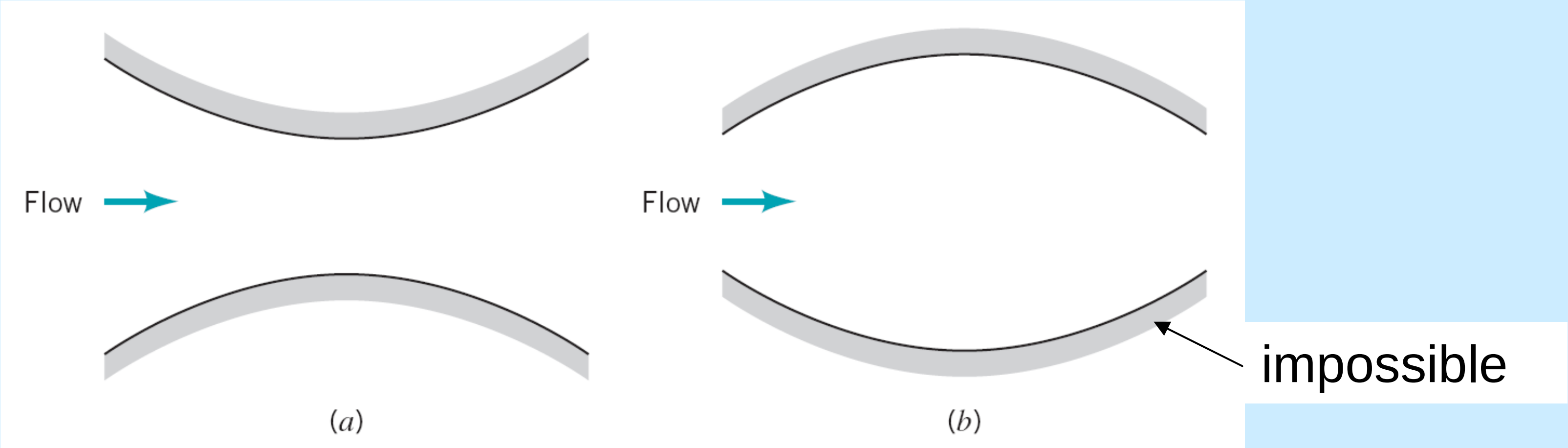
$$\frac{d\rho}{\rho}+\frac{dA}{A}=\frac{dA}{A}\frac{1}{1-\mathrm{Ma}^2} \quad \Rightarrow \quad \frac{d\rho}{\rho}=\frac{dA}{A}\frac{\mathrm{Ma}^2}{1-\mathrm{Ma}^2} \qquad (11.49)$$

(11.49): For subsonic flow, density and area changes are in the same direction; for supersonic flow, density and area changes are in the opposite direction.

$$\text{From (11.48):} \quad \frac{dA}{dV}=-\frac{A}{V}(1-\mathrm{Ma}^2)$$

$$\text{When } \mathrm{Ma}=1 \rightarrow \frac{dA}{dV}=0 \Rightarrow$$

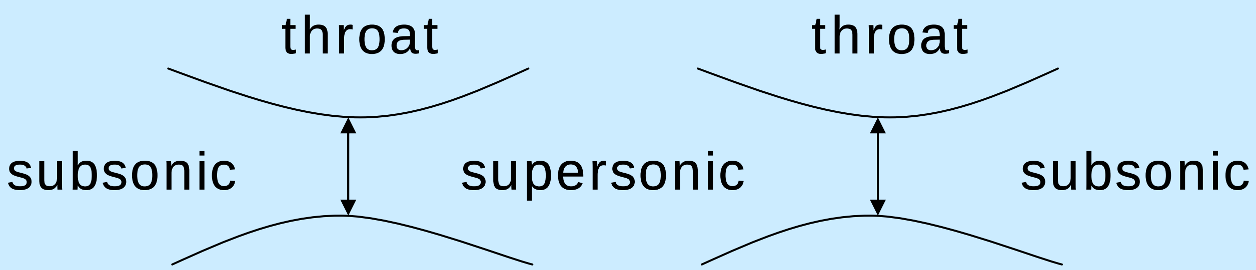
The area associated with Ma=1 is either a minimum or a maximum.



(a) (b)

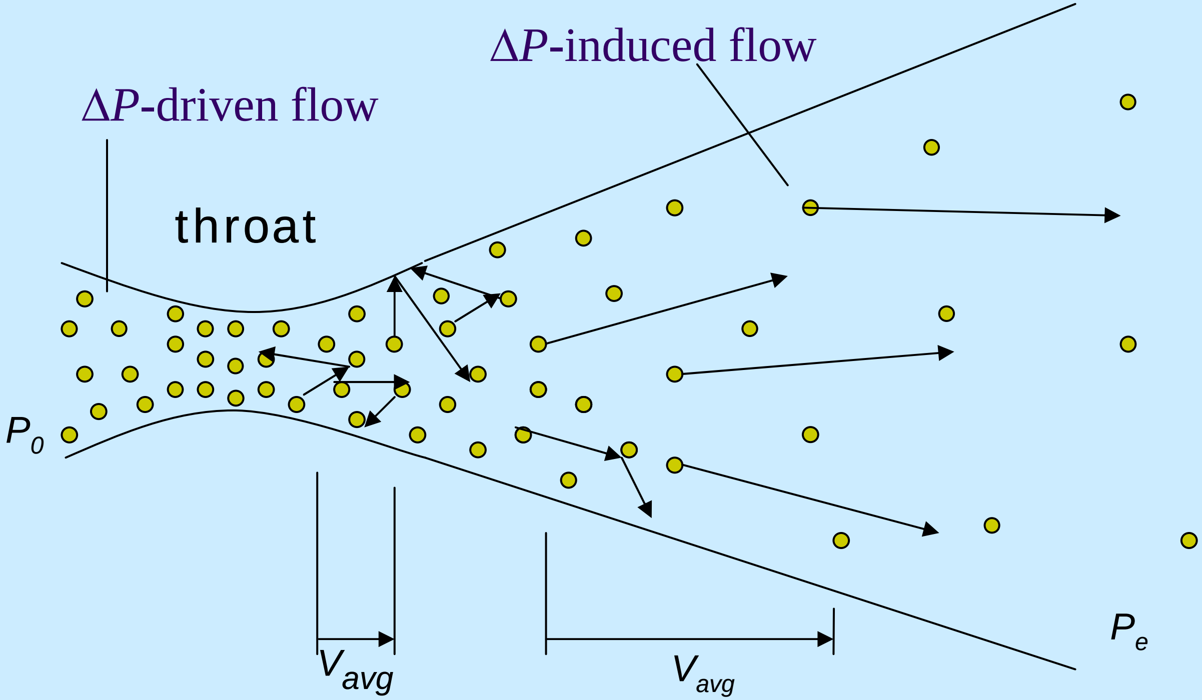
Therefore the sonic conduction Ma=1 can be obtained in a converging-diverging duct at the minimum area location.

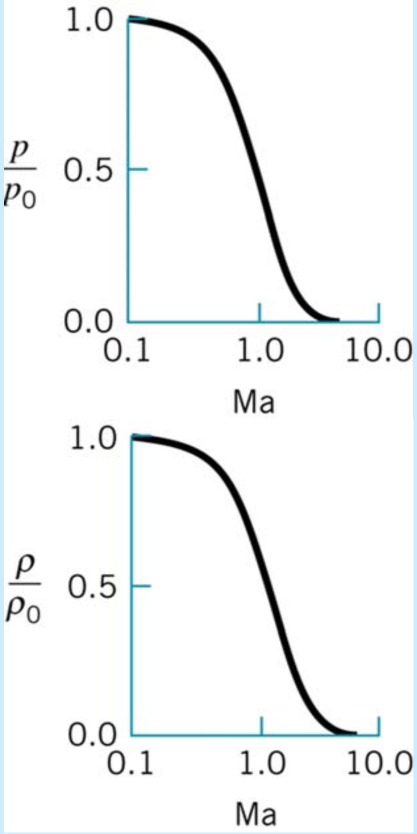
- For subsonic flow → converging diverging nozzle
- For supersonic flow → converging diverging diffuser



Converging-diverging nozzle Converging-diverging diffuser

What is the **physical** reason that supersonic flow develops in the diverging duct as long as sonic flow is reached at the throat?





Why can V only be accelerated to c in the converging duct, no matter how low the back pressure P_e is?

11.4.2 Converging–Diverging Duct Flow

- For an isentropic flow

$$\frac{p}{\rho^k} = \text{constant} = \frac{p_0}{\rho_0^{\ k}}$$

- streamwise equation of motion for steady, frictionless flow

$$\frac{dp}{\rho} + d\left(\frac{V^2}{2}\right) = 0, \quad \gamma dz \text{ neglected}$$

$$\frac{p_0^{1/k}}{\rho_0}\frac{dp}{p^{1/k}} + d\left(\frac{V^2}{2}\right) = 0 \qquad \left(\because \frac{p}{\rho^k} = \frac{p_0}{\rho_0^k} \rightarrow \rho = p^{1/k}\Big/\left(p_0^{1/k}/\rho_0\right)\right)$$

$$\frac{k}{k-1}\frac{p_0^{1/k}}{\rho_0}\left[p_0^{\frac{k-1}{k}} - p^{\frac{k-1}{k}}\right] - \frac{V^2}{2} = 0$$

$$\frac{k}{k-1}\left[\frac{p_0}{\rho_0} - \frac{p}{\rho}\right] - \frac{V^2}{2} = 0$$

-For an idea gas

$$\frac{p_0}{\rho_0} = RT_0, \quad \frac{p}{\rho} = RT$$

$$\therefore \frac{kR}{k-1}[T_0 - T] - \frac{V^2}{2} = 0 \ \text{ or } \ c_p(T_0 - T) - \frac{V^2}{2} = 0 \quad (\because c_p = \frac{kR}{k-1})$$

$$\Rightarrow h_0 - (h + \frac{V^2}{2}) = 0$$ where h_0 is the stagnation enthalpy

$$\frac{kRT_0}{k-1} = \frac{kRT}{k-1} + \frac{V^2}{2} \qquad kRT_0 = kRT + \frac{k-1}{2}V^2$$

$$\frac{T_0}{T} = 1 + \frac{k-1}{2}\frac{V^2}{kRT} = 1 + \frac{k-1}{2}\mathrm{Ma}^2 \quad (kRT = c^2)$$

$$\frac{T}{T_0} = \frac{1}{1 + [(k-1)/2]\mathrm{Ma}^2} \qquad (11.56)$$

With $\dfrac{p}{\rho} = RT$

$$\frac{p}{p_0}\frac{\rho_0}{\rho} = \frac{T}{T_0} \quad \left(\because \frac{p}{\rho^k} = \frac{p_0}{\rho_0^k} \Rightarrow \frac{\rho_0}{\rho} = (\frac{p_0}{p})^{\frac{1}{k}} \right)$$

$$\frac{p}{p_0}(\frac{p}{p_0})^{-\frac{1}{k}} = \frac{T}{T_0} \quad \rightarrow \quad (\frac{p}{p_0})^{\frac{k-1}{k}} = \frac{T}{T_0} \quad \Rightarrow \quad \frac{p}{p_0} = (\frac{T}{T_0})^{\frac{k}{k-1}}$$

$$\therefore \frac{p}{p_0} = [\frac{1}{1+[(k-1)/2]\mathrm{Ma}^2}]^{\frac{k}{k-1}} \qquad (11.59)$$

using isentropic relation

$$\frac{\rho}{\rho_0} = [\frac{1}{1+[(k-1)/2]\mathrm{Ma}^2}]^{\frac{1}{k-1}} \qquad (11.60)$$

$$\frac{T}{T_0} = \frac{1}{1+[(k-1)/2]\mathrm{Ma}^2} \qquad (11.56)$$

$$\frac{p}{p_0} = \left[\frac{1}{1+[(k-1)/2]\mathrm{Ma}^2}\right]^{\frac{k}{k-1}}$$

$$\frac{\rho}{\rho_0} = \left[\frac{1}{1+[(k-1)/2]\mathrm{Ma}^2}\right]^{\frac{1}{k-1}}$$

$$\frac{T}{T_0} = \frac{1}{1+[(k-1)/2]\mathrm{Ma}^2}$$

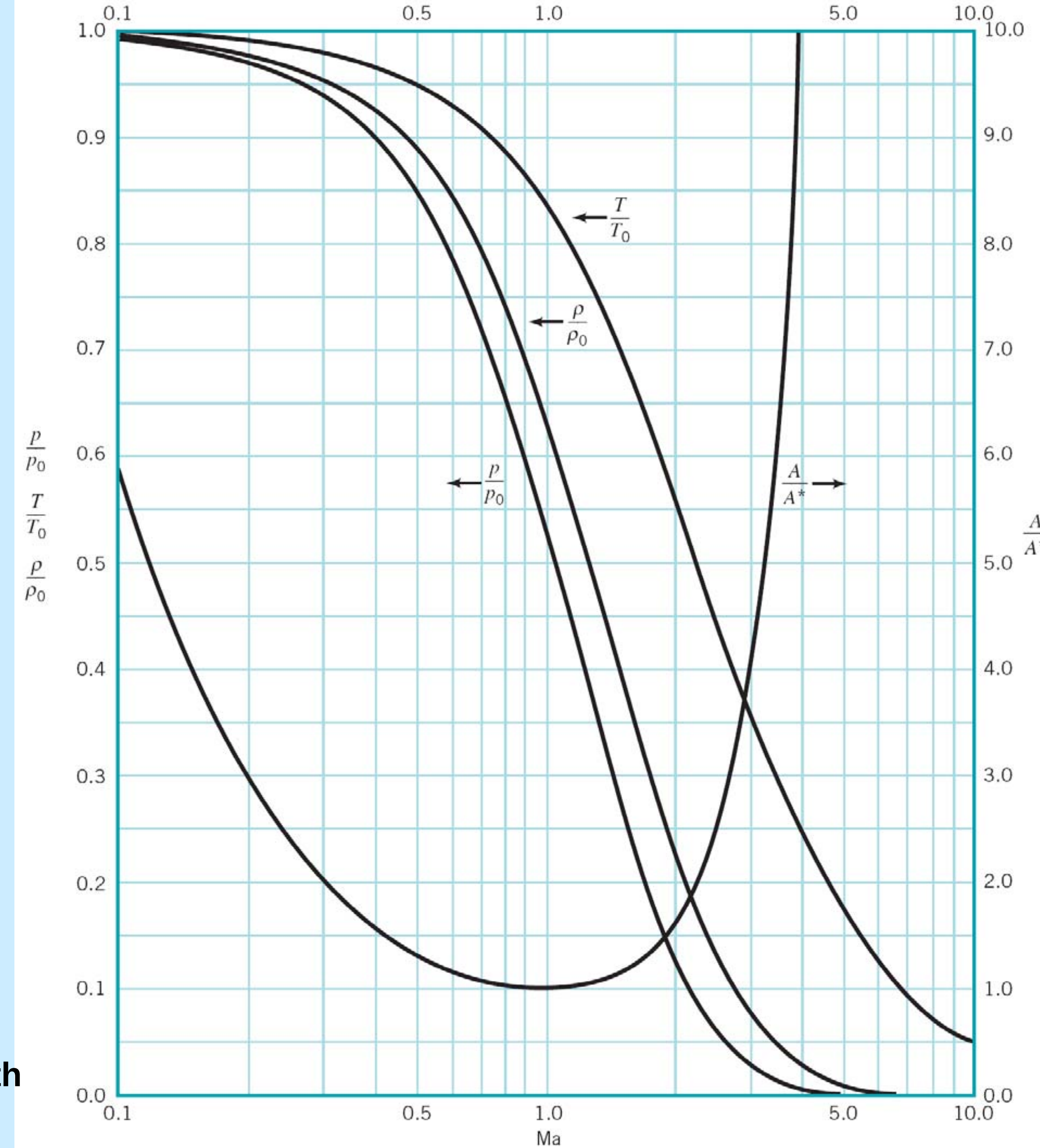
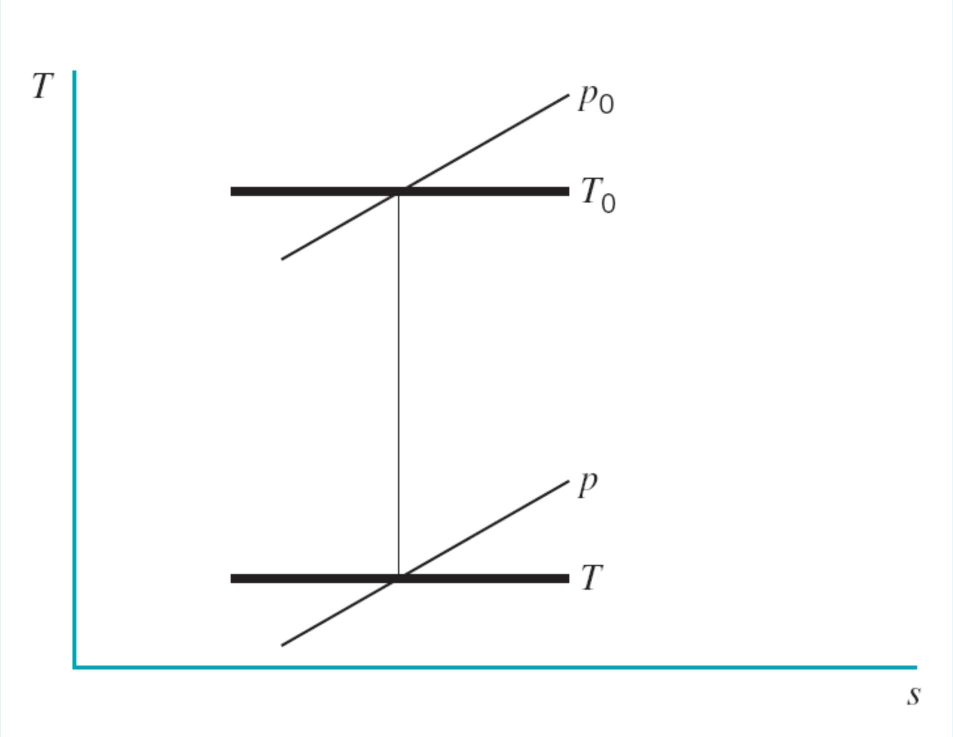


Figure D1 (p. 718)

Isentropic flow of an ideal gas with *k* = 1.4.



$$c_p T_0 = c_p T + \frac{V^2}{2}$$

Figure 11.7
The (*T* – *s*) diagram relating stagnation and static states.

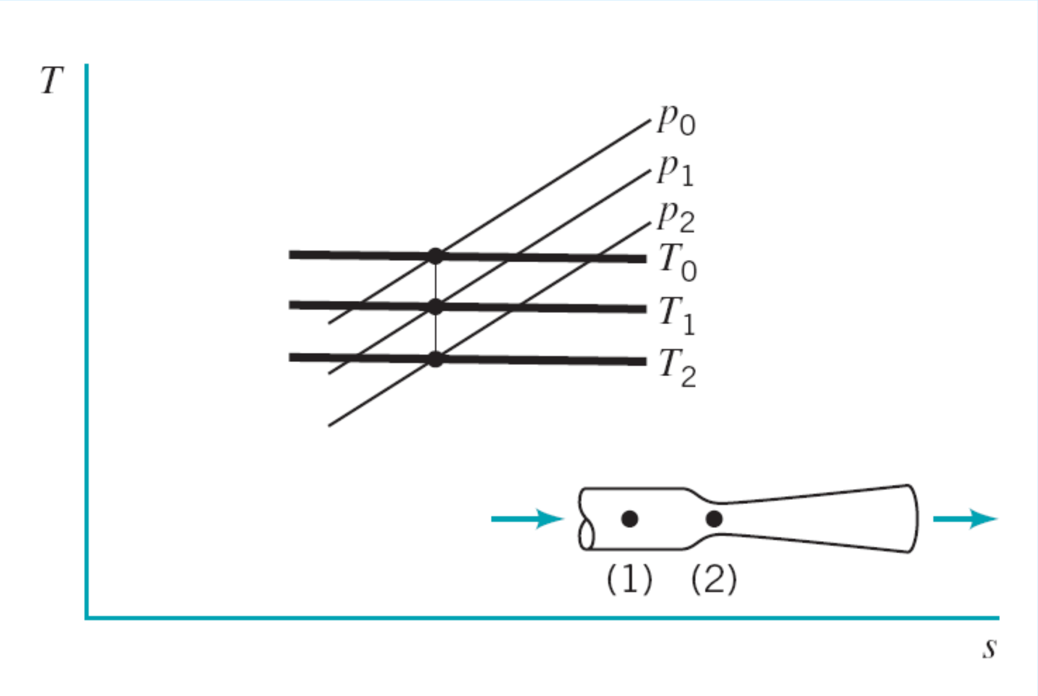
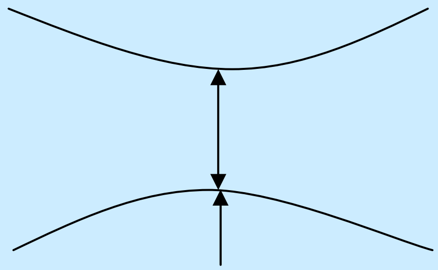
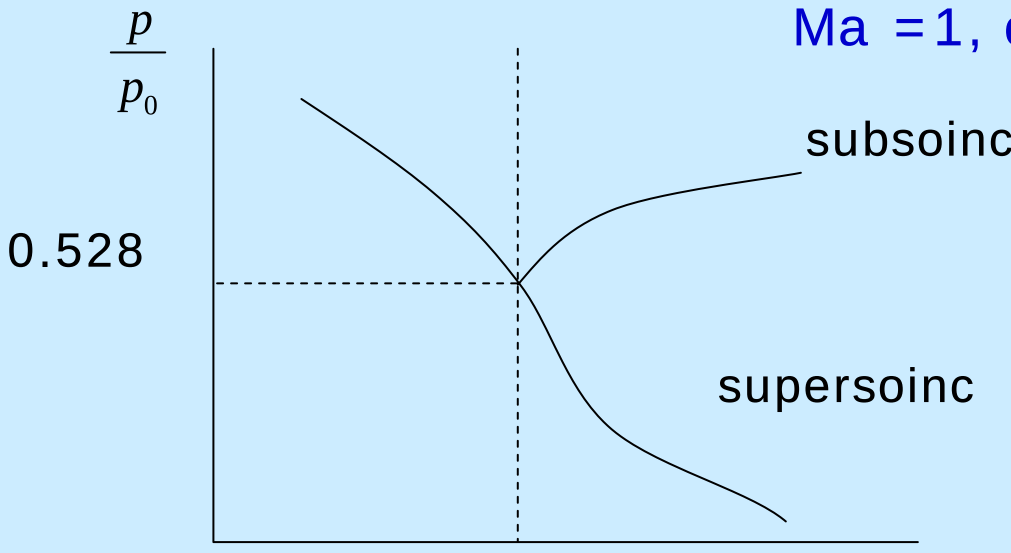


Figure 11.8
The *T* – *s* diagram for Venturi meter flow.

- Any further decrease of the back pressure will not effect the flow in the converging portion of the duct.
- At Ma=1 the information about pressure can not move upstream
- Consider the **choked flow** where at the throat Ma=1, the state is called **critical state**



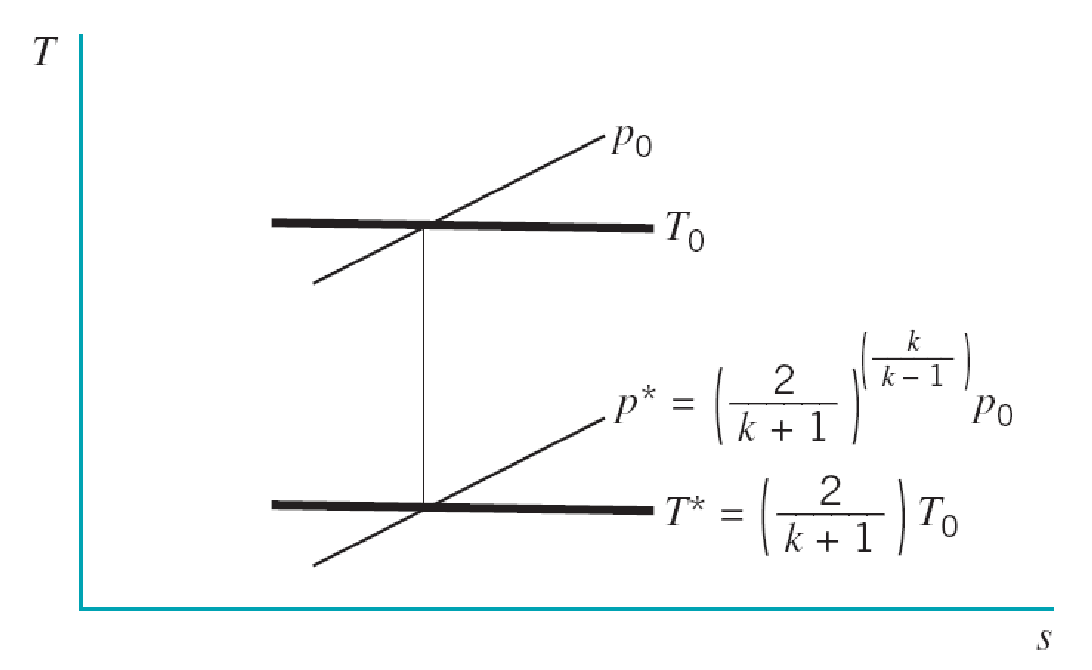
Ma =1, critical state (choked flow)



Critical State:

Set Ma=1 in (11.56), (11.59), (11.60)

$$\frac{p^*}{p_0} = \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}}$$



For k=1.4

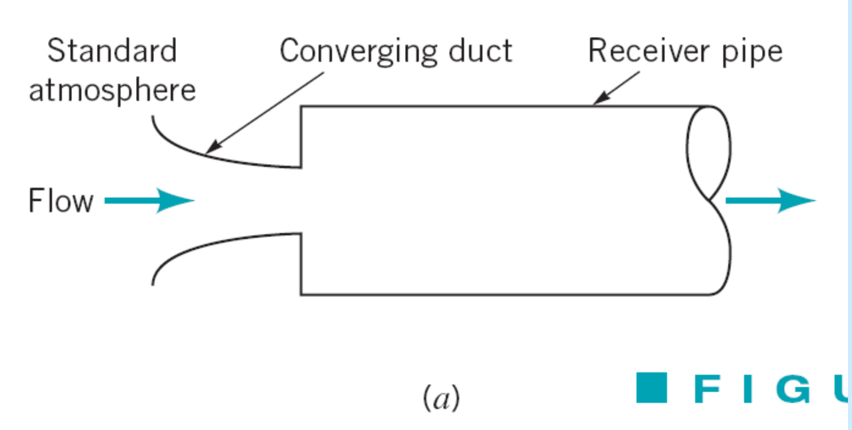
$$\left(\frac{p^*}{p_0}\right)_{k=1.4} = 0.528$$

$$p^*_{k=1.4} = 0.528\, p_{\text{atm}}$$

$$\frac{T^*}{T_0} = \frac{2}{k+1} \rightarrow \left(\frac{T^*}{T_0}\right)_{k=1.4} = 0.833 \text{ or } T^*_{k=1.4} = 0.833T_0 = 0.833T_{\text{atm}}$$

$$\frac{\rho^*}{\rho_0} = \frac{p^*}{T^*}\frac{T_0}{p_0} = \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}}\left(\frac{k+1}{2}\right) = \left(\frac{2}{k+1}\right)^{\frac{1}{k-1}} \rightarrow \left(\frac{\rho^*}{\rho_0}\right)_{k=1.4} = 0.643$$

Example 11.5



$p_0 = 101\ \text{kPa}$

$\rho_0 = 1.23\ \text{kg/m}^3$

$T_0 = 288\ \text{K}$

Find $\dot{m} =$ (a)80 kPa, (b)40 kPa.

Critical pressure $p^* = 0.528 p_0 = 53.3$ kPa

(a) $p_a > p^*$ ∴.the throat is not choked

$$\frac{p}{p_0} = \frac{80}{101} = \left[\frac{1}{1+[(k-1)/2]\text{Ma}^2}\right]^{\frac{k}{k-1}} \rightarrow \text{Ma}_{\text{th}} = 0.587$$

$$\frac{\rho}{\rho_0} = \left[\frac{1}{1+[(k-1)/2]\text{Ma}^2}\right]^{\frac{1}{k-1}} \qquad \rho_0 = 1.23\ kg/m^3 \quad \Rightarrow \quad \rho = 1.04\ kg/m^3$$

$$\frac{T}{T_0} = \frac{1}{1+[(k-1)/2]\text{Ma}^2} \qquad \Rightarrow T = 269\ \text{K}$$

$$V = \text{Ma} * \sqrt{kRT} \qquad \Rightarrow V = 193\ \text{m/s}$$

$$\dot{m} = \rho V A = 0.0201\ \text{kg/s}$$

(b) p_b=40 kPa < p^*=53.3 the flow is choked at the throat Ma = 1

$$\frac{\rho}{\rho_0}=\left[\frac{1}{1+[(k-1)/2]\mathrm{Ma}^2}\right]^{\frac{1}{k-1}} \Rightarrow \rho = 0.634\rho_0 = 0.78\ \mathrm{kg/m^3}$$

$$\frac{T}{T_0}=\frac{1}{1+[(k-1)/2]\mathrm{Ma}^2} \Rightarrow T = 240\ K$$

$$V=\mathrm{Ma}*\sqrt{kRT} \Rightarrow V = 310\ \mathrm{m/s}$$

$$\dot{m}=\rho VA = 0.0242\ \mathrm{kg/s}$$

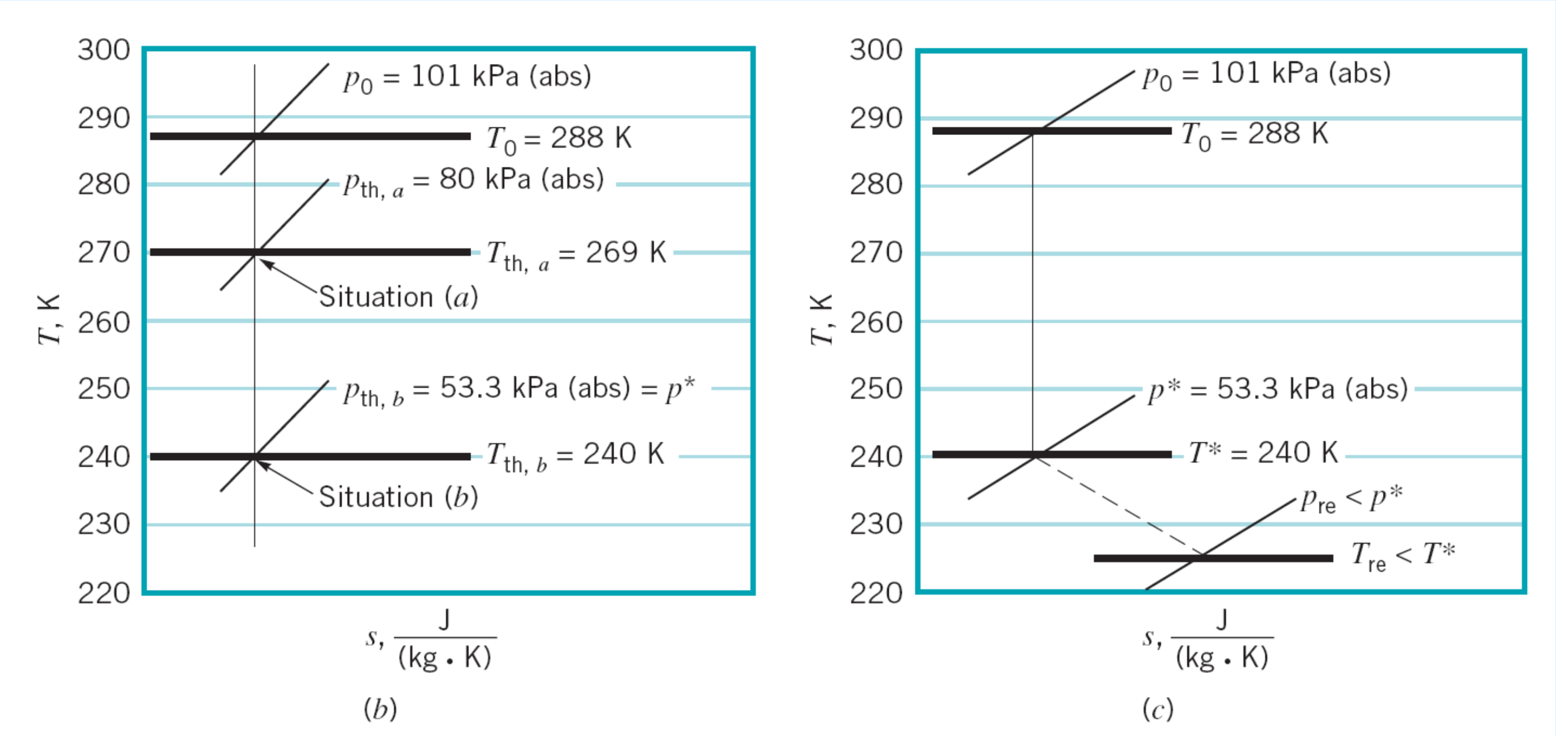
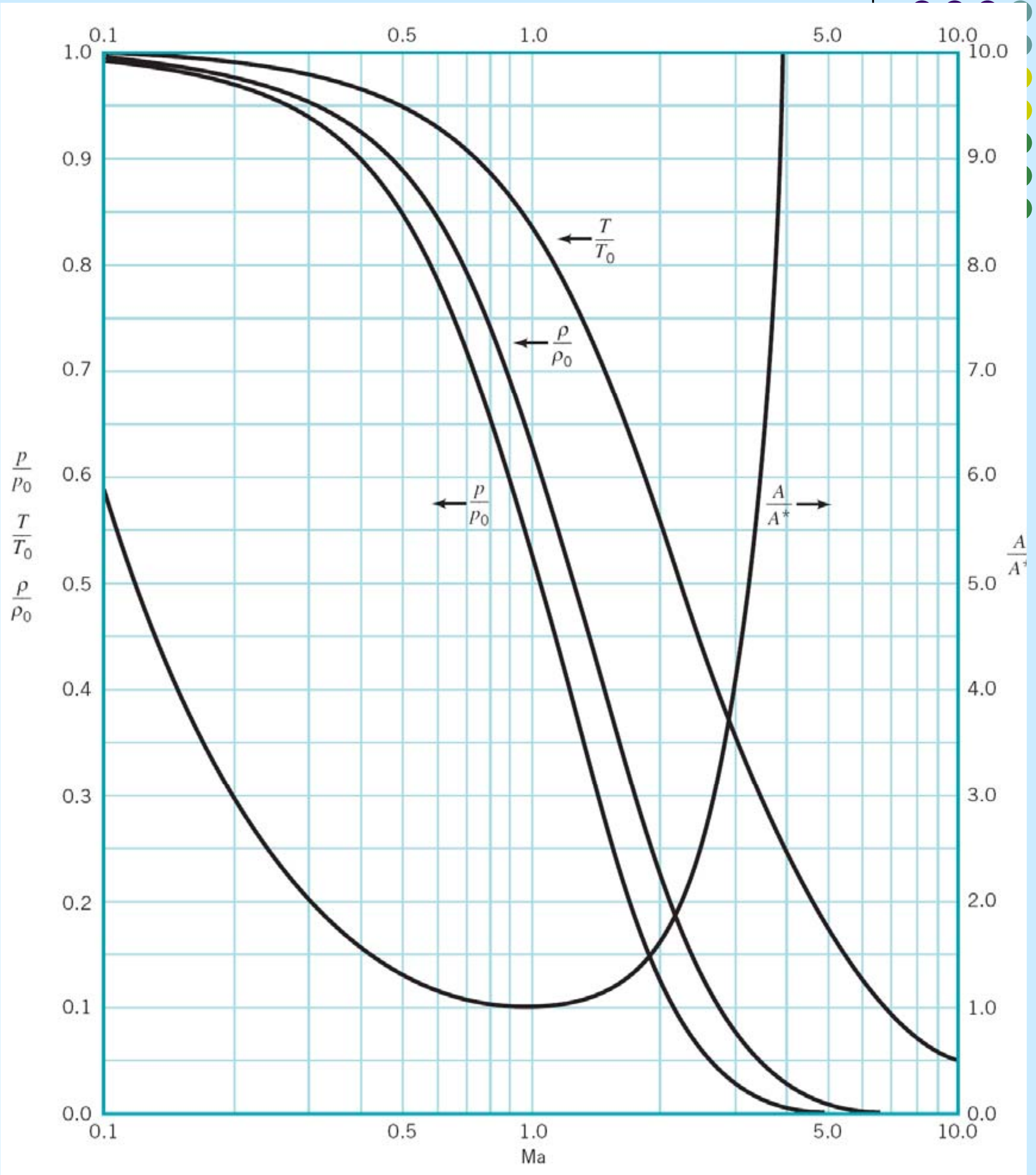


Figure D1 (p. 718)

Isentropic flow of an ideal gas with *k* = 1.4. (Graph provided by Dr. Bruce A. Reichert.)



Ratio A/A^*

$$\rho AV = \rho^* A^* V^* \quad \text{or} \quad \frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{V^*}{V}$$

$$\text{with } V^* = \sqrt{kRT^*} \quad \text{and} \quad V = \text{Ma}\sqrt{kRT}$$

$$\rightarrow \frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{\sqrt{kRT^*}}{\sqrt{kRT}\,\text{Ma}} = \frac{1}{\text{Ma}}\frac{\rho^*}{\rho_0}\sqrt{\frac{T^*/T_0}{T/T_0}}$$

$$= \frac{1}{\text{Ma}}\left(\frac{2}{k+1}\right)^{\frac{1}{k-1}}\left[1+\frac{k+1}{2}\text{Ma}^2\right]^{\frac{1}{k-1}}\left[\frac{2/k+1}{1/1+[(k-1)/2]\text{Ma}^2}\right]^{\frac{1}{2}}$$

$$\Rightarrow \frac{A}{A^*} = \frac{1}{\text{Ma}}\left[\frac{1+[(k-1)/2]\text{Ma}^2}{1+[(k-1)/2]}\right]^{\frac{k+1}{2(k-1)}}$$

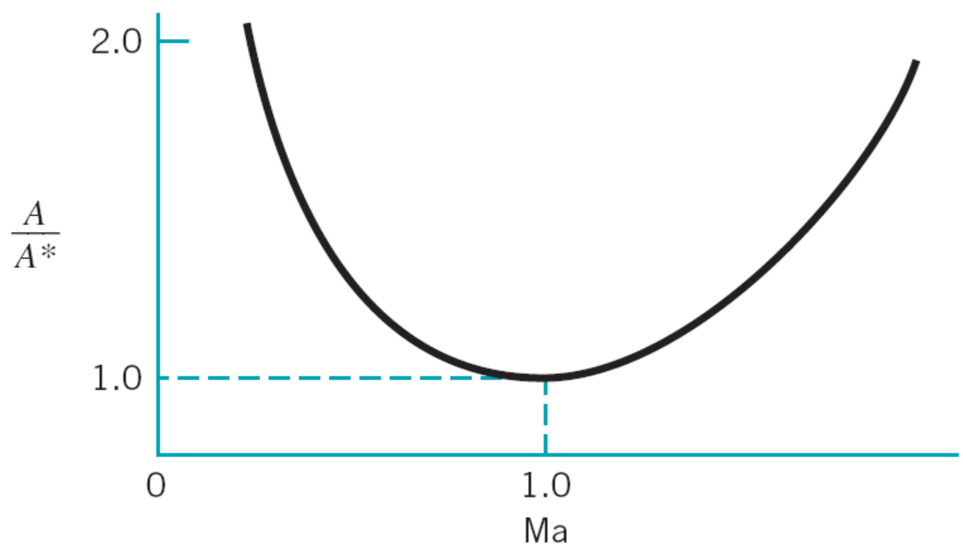
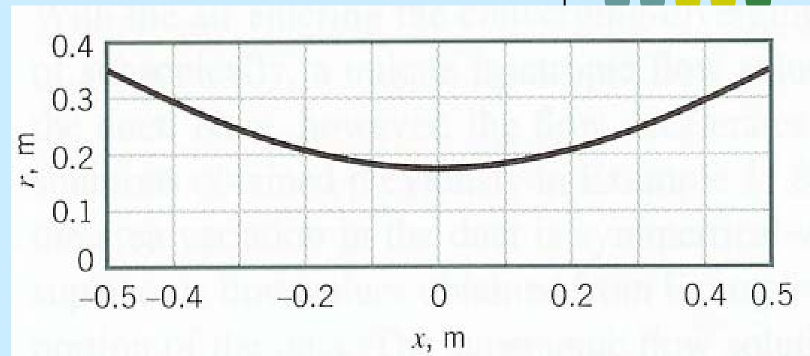


Figure 11.10 (p. 639)

The variation of area ratio with Mach number for isentropic flow of an ideal gas (k = 1.4, linear coordinate scales).

Example 11.8 Air entering subsonically at standard atmospheric condition, A=0.1+x^2

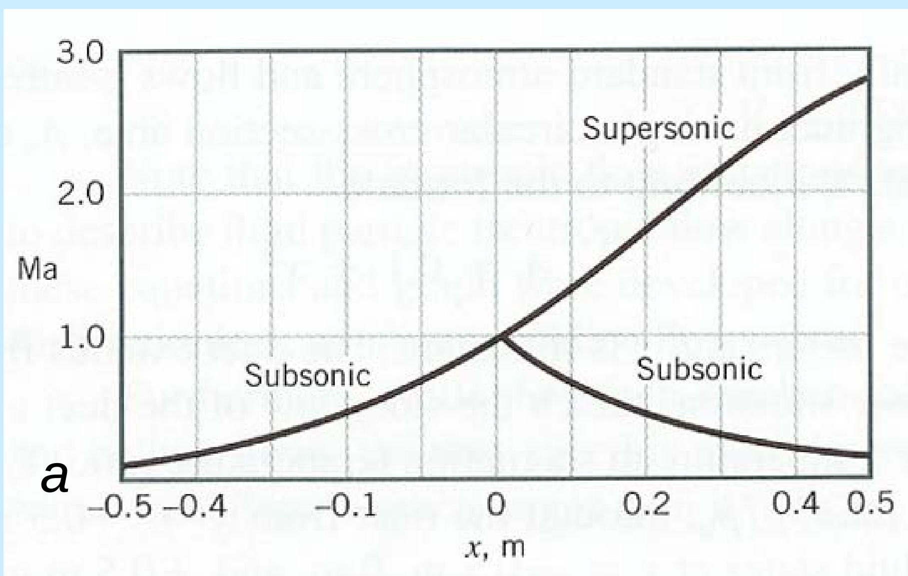


$$A = \pi r^2 \rightarrow r = (\frac{A}{\pi})^{\frac{1}{2}} = (\frac{0.1 + x^2}{\pi})^{\frac{1}{2}}$$

For the flow to be chocked

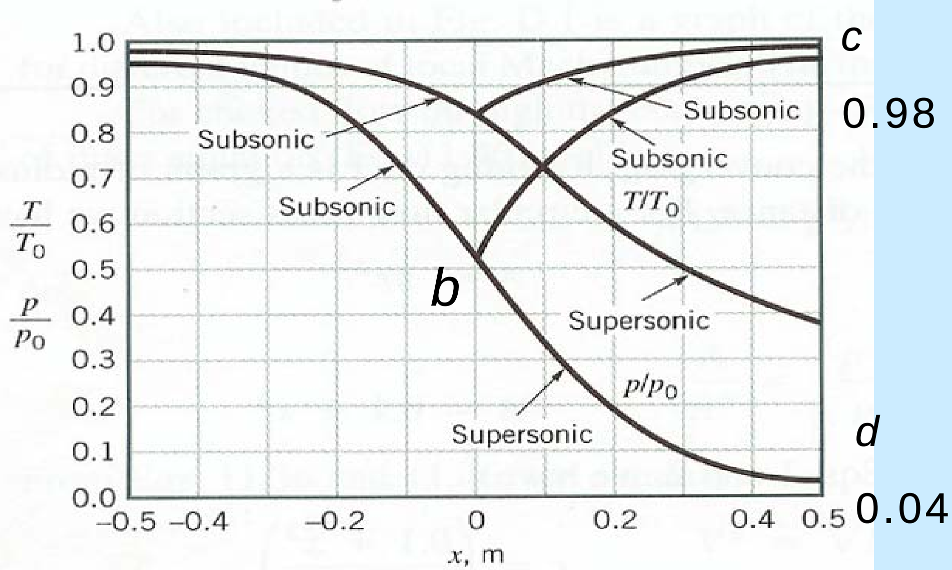
At throat, $x = 0 \rightarrow A^* = 0.1$

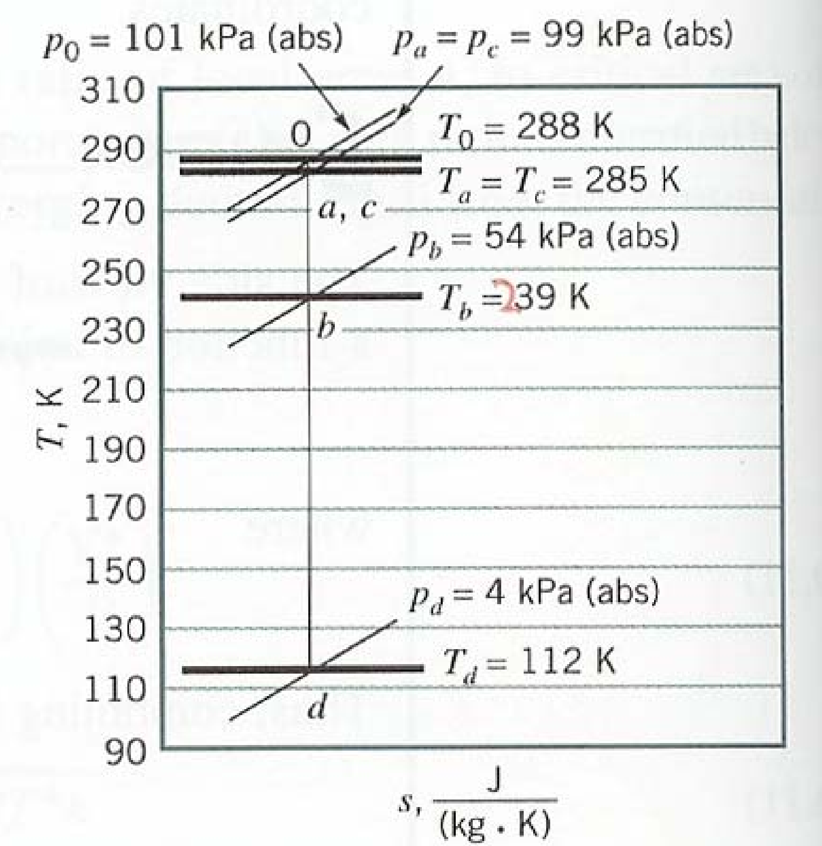
$$\frac{A}{A^*} = \frac{0.1 + x^2}{0.1}, \quad \text{using (11.71)} \quad \Rightarrow \text{Ma} \quad \Rightarrow \frac{p}{p_0}, \quad \frac{T}{T_0}$$



d

c





x (m)	From Eq. 3, r (m)	From Eq. 5, A/A^*	From Fig. D.1 Ma	From Fig. D.1 T/T_0	From Fig. D.1 p/p_0	State
Subsonic Solution						
−0.5	0.334	3.5	0.17	0.99	0.98	*a*
−0.4	0.288	2.6	0.23	0.99	0.97	
−0.3	0.246	1.9	0.32	0.98	0.93	
−0.2	0.211	1.4	0.47	0.96	0.86	
−0.1	0.187	1.1	0.69	0.91	0.73	
0	0.178	1	1.00	0.83	0.53	*b*
+0.1	0.187	1.1	0.69	0.91	0.73	
+0.2	0.211	1.4	0.47	0.96	0.86	
+0.3	0.246	1.9	0.32	0.98	0.93	
+0.4	0.288	2.6	0.23	0.99	0.97	
+0.5	0.344	3.5	0.17	0.99	0.98	*c*
Supersonic Solution						
+0.1	0.187	1.1	1.37	0.73	0.33	
+0.2	0.211	1.4	1.76	0.62	0.18	
+0.3	0.246	1.9	2.14	0.52	0.10	
+0.4	0.288	2.6	2.48	0.45	0.06	
+0.5	0.334	3.5	2.80	0.39	0.04	*d*

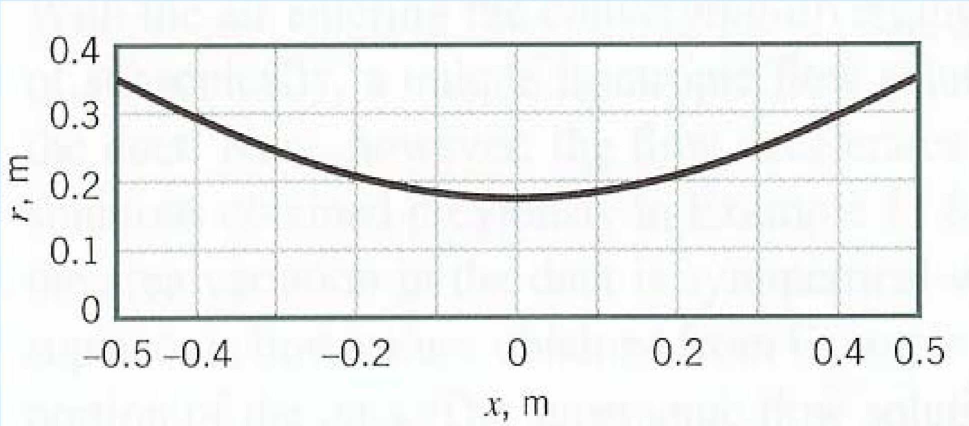
of Ma after A/A^* values of about 10. The ratio of p/p_0 becomes vanishingly small suggesting a practical limit to the expansion.

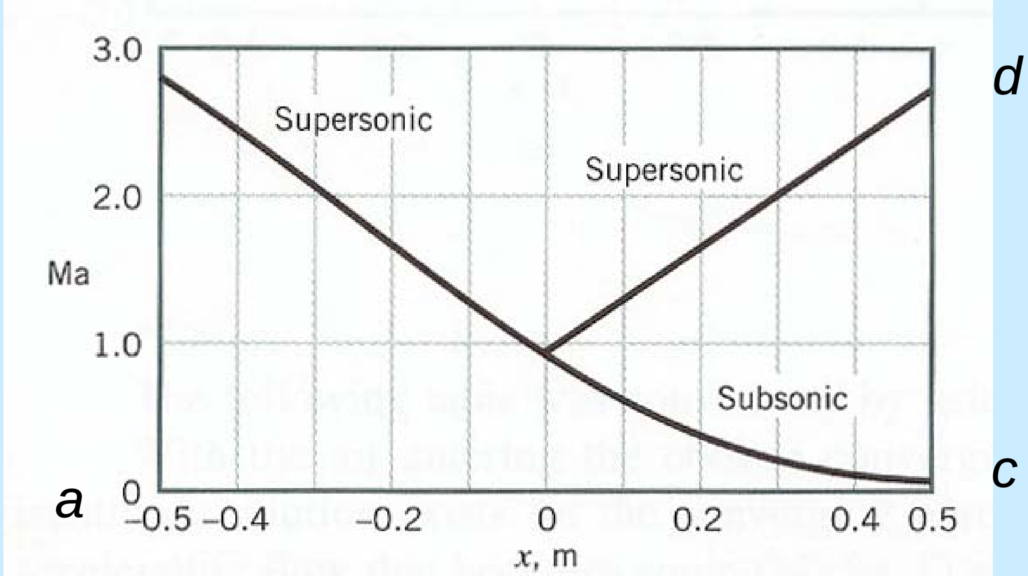
Example 11.9 Air entering supersonically at standard atmospheric condition, $A=0.1+x^2$

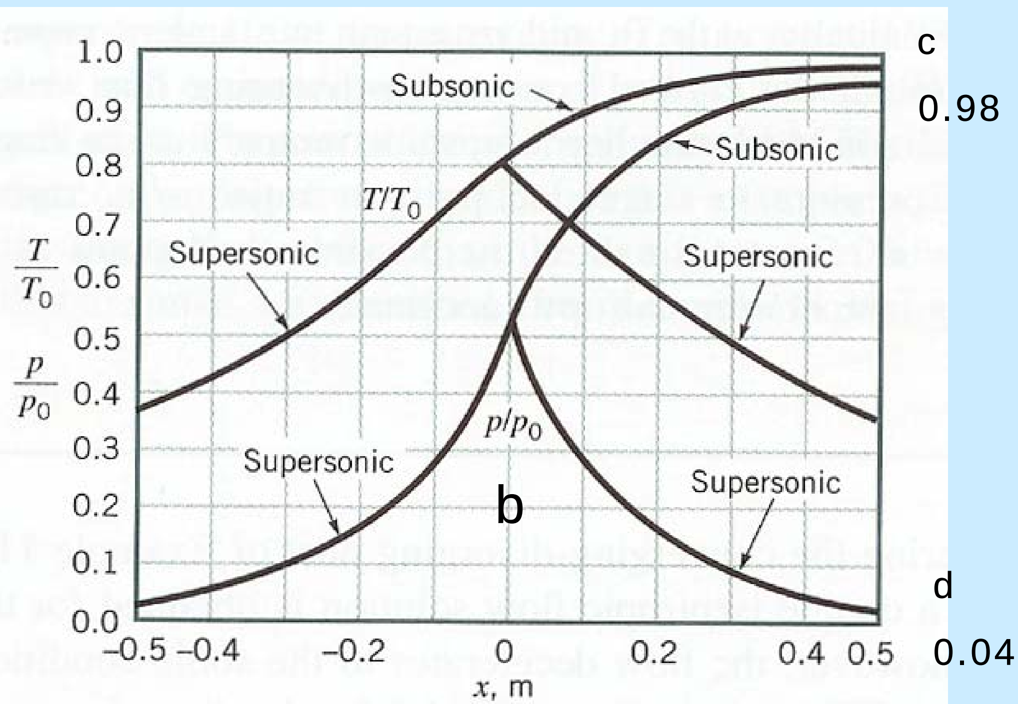
For the flow to be chocked

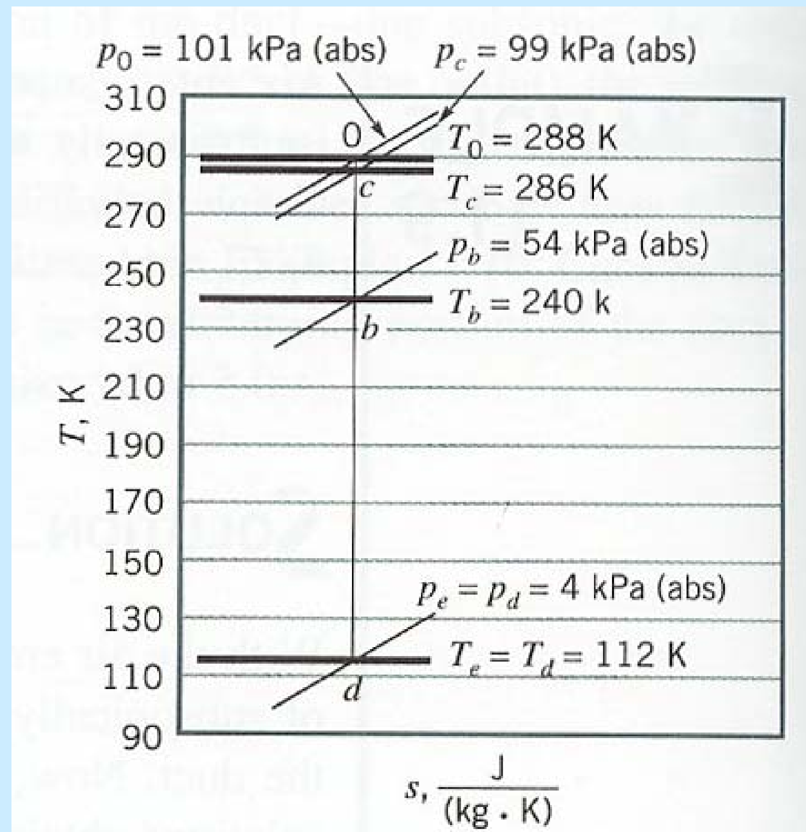
$$\text{At throat, } x=0 \rightarrow A^* = 0.1$$

$$\frac{A}{A^*} = \frac{0.1+x^2}{0.1}, \quad \text{using (11.71)} \quad \Rightarrow \text{Ma} \quad \Rightarrow \frac{p}{p_0}, \quad \frac{T}{T_0}$$




3.0
2.0
1.0
0
Ma
Supersonic
Supersonic
Subsonic
a
d
c
−0.5 −0.4
−0.2
0
0.2
0.4 0.5
x, m


1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
T/T0
p/p0
Subsonic
Subsonic
Supersonic
Supersonic
Supersonic
Supersonic
b
c
0.98
d
0.04
−0.5 −0.4
−0.2
0
0.2
0.4 0.5
x, m

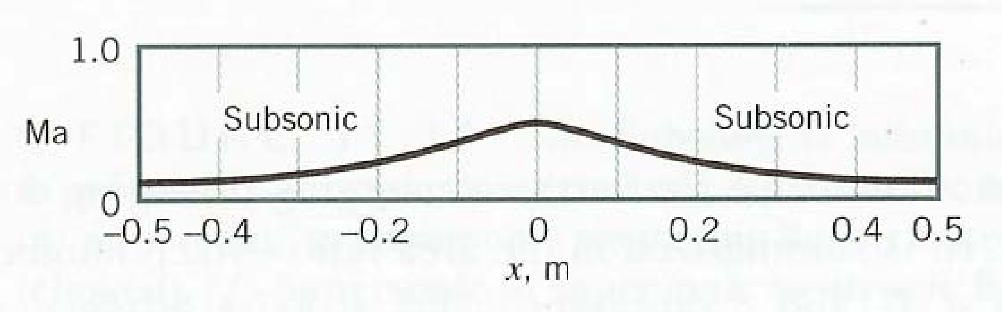

p0 = 101 kPa (abs)
pc = 99 kPa (abs)
T0 = 288 K
Tc = 286 K
pb = 54 kPa (abs)
Tb = 240 k
pe = pd = 4 kPa (abs)
Te = Td = 112 K
0
c
b
d
310
290
270
250
230
210
190
170
150
130
110
90
T, K
s, J/(kg · K)

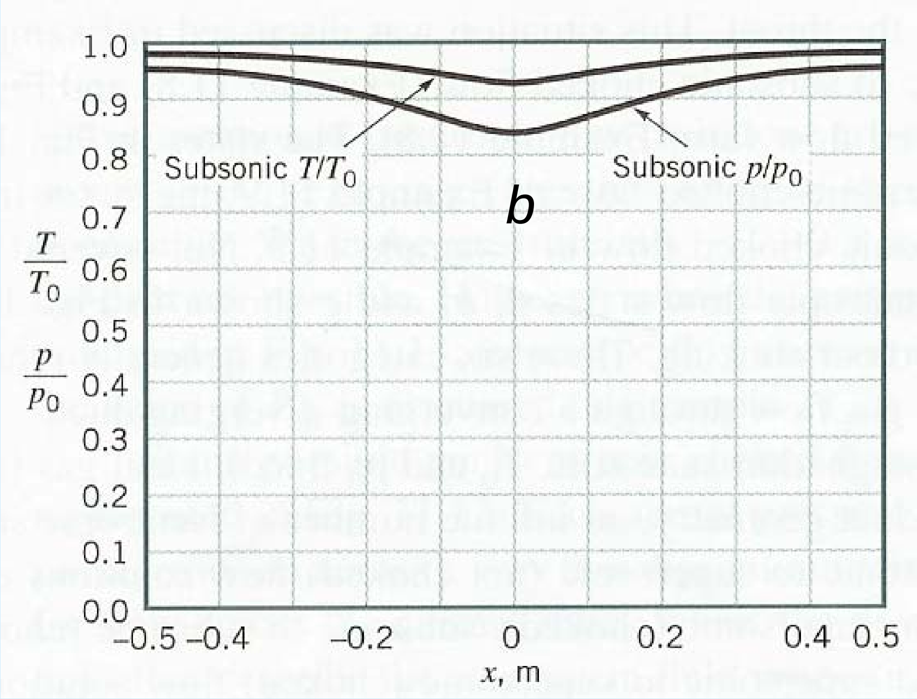
Example 11.10 Ma=0.48 at throat, A=0.1+x^2

The Ma at throat is 0.48

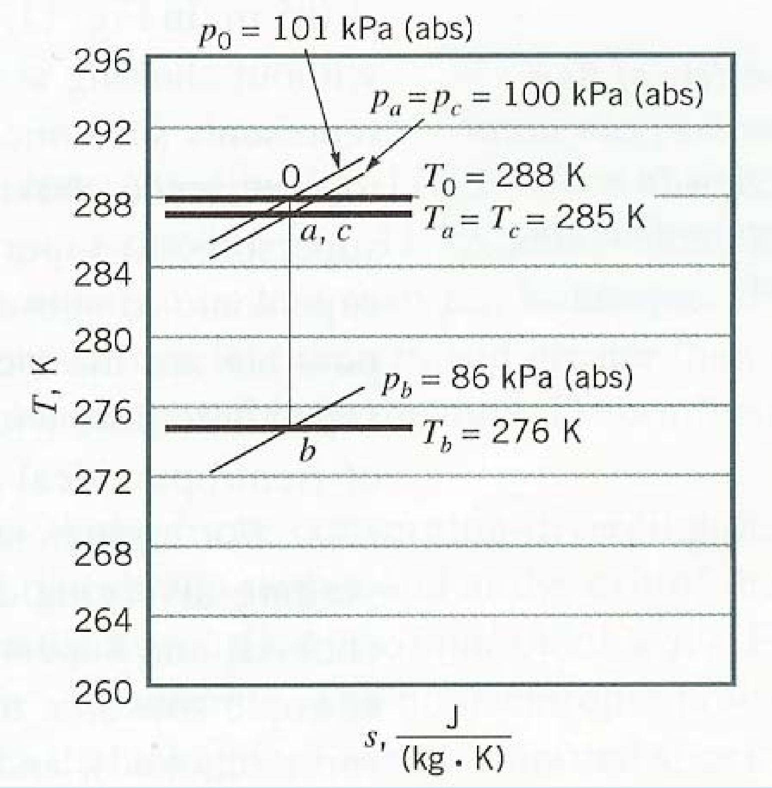
$$Ma = 0.48 \Rightarrow \frac{p}{p_0}, \quad \frac{T}{T_0}, \quad \frac{A}{A^*}$$

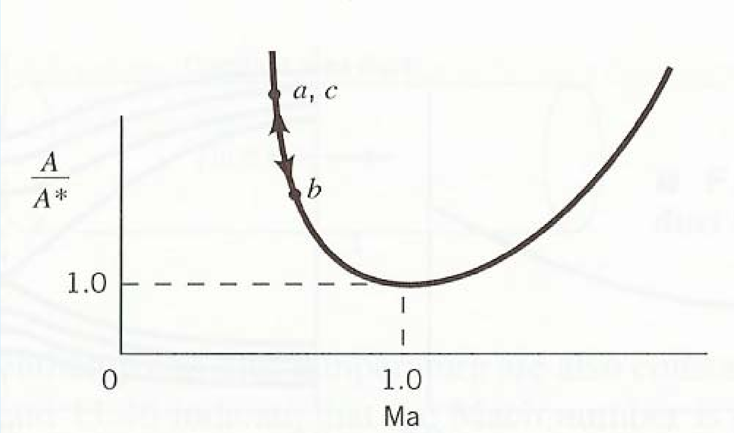
$$\frac{A}{A^*} = \frac{0.1}{A^*} = 1.4 \qquad \Rightarrow A^* = 0.07$$



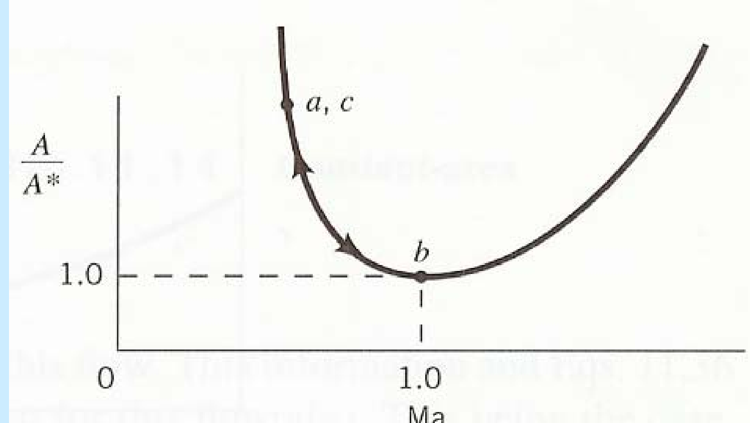


c

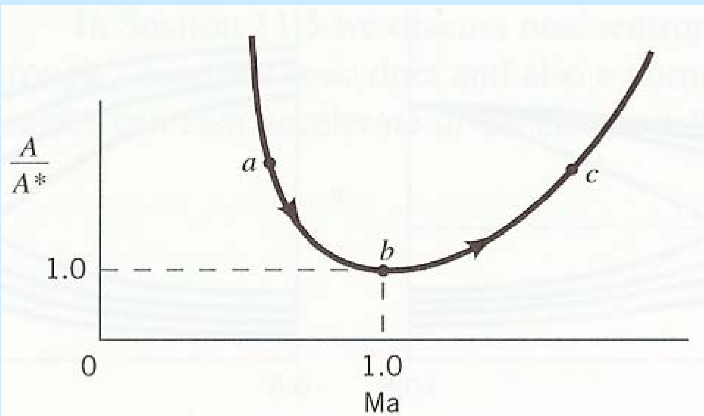




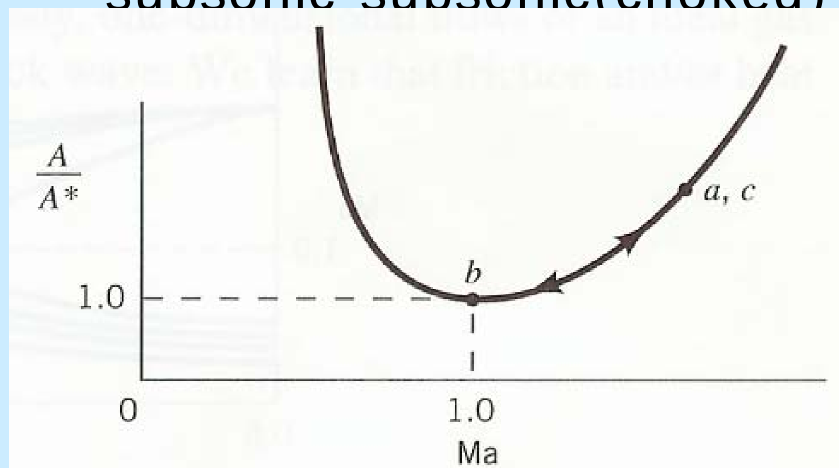
subsonic-subsonic



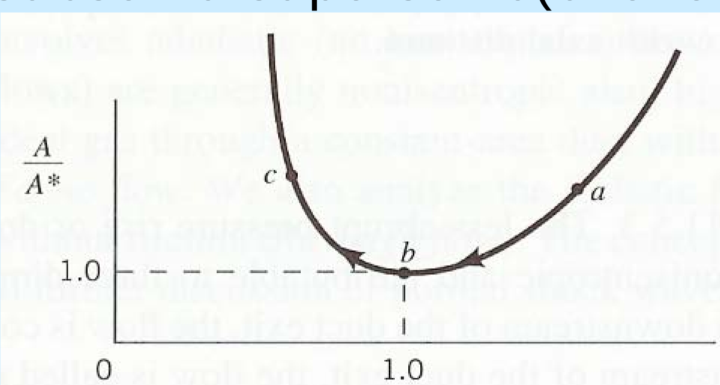
subsonic-subsonic(choked)



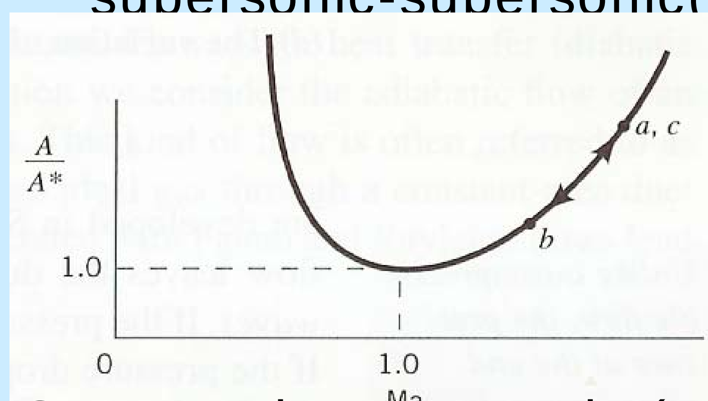
subsonic-supersonic(choked)



supersonic-supersonic(choked)

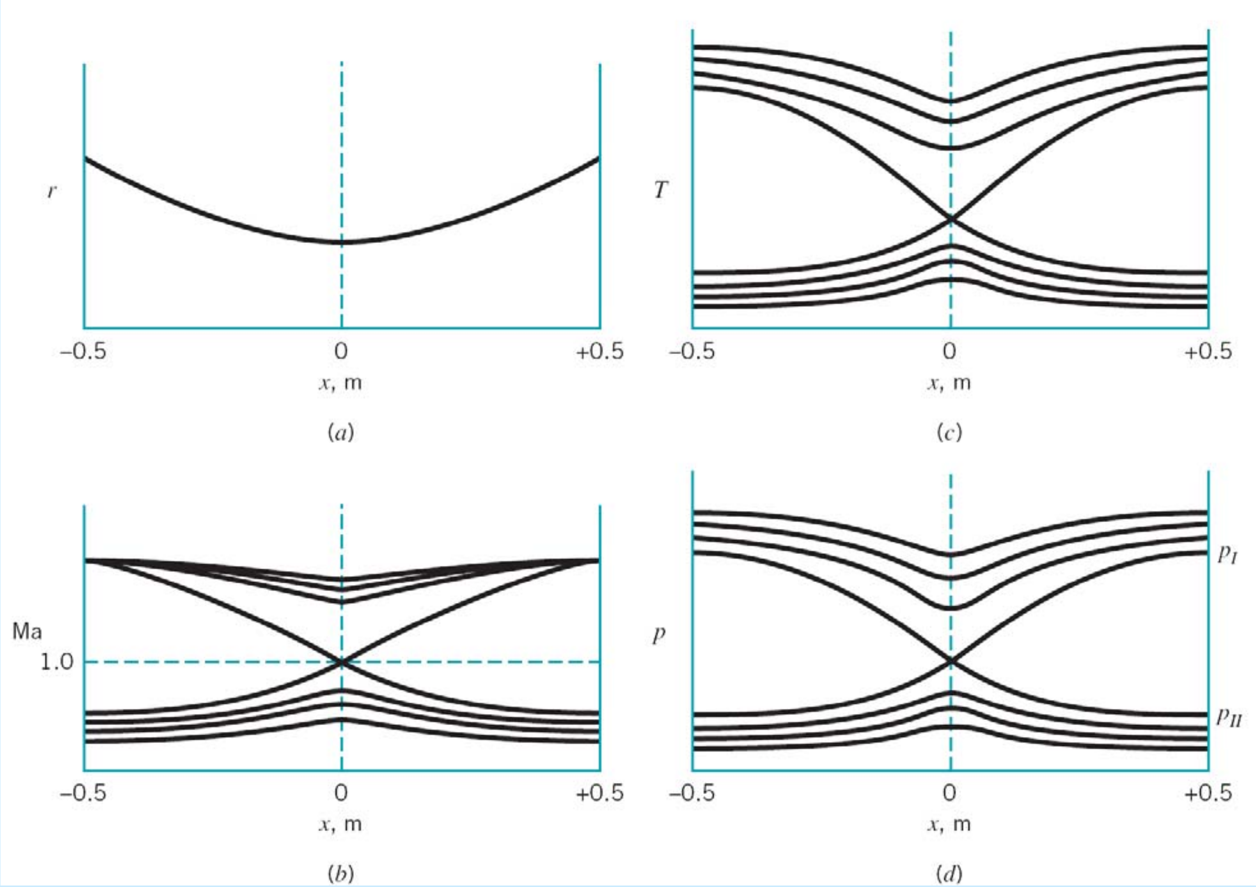


Supersonic-subsonic (choked)



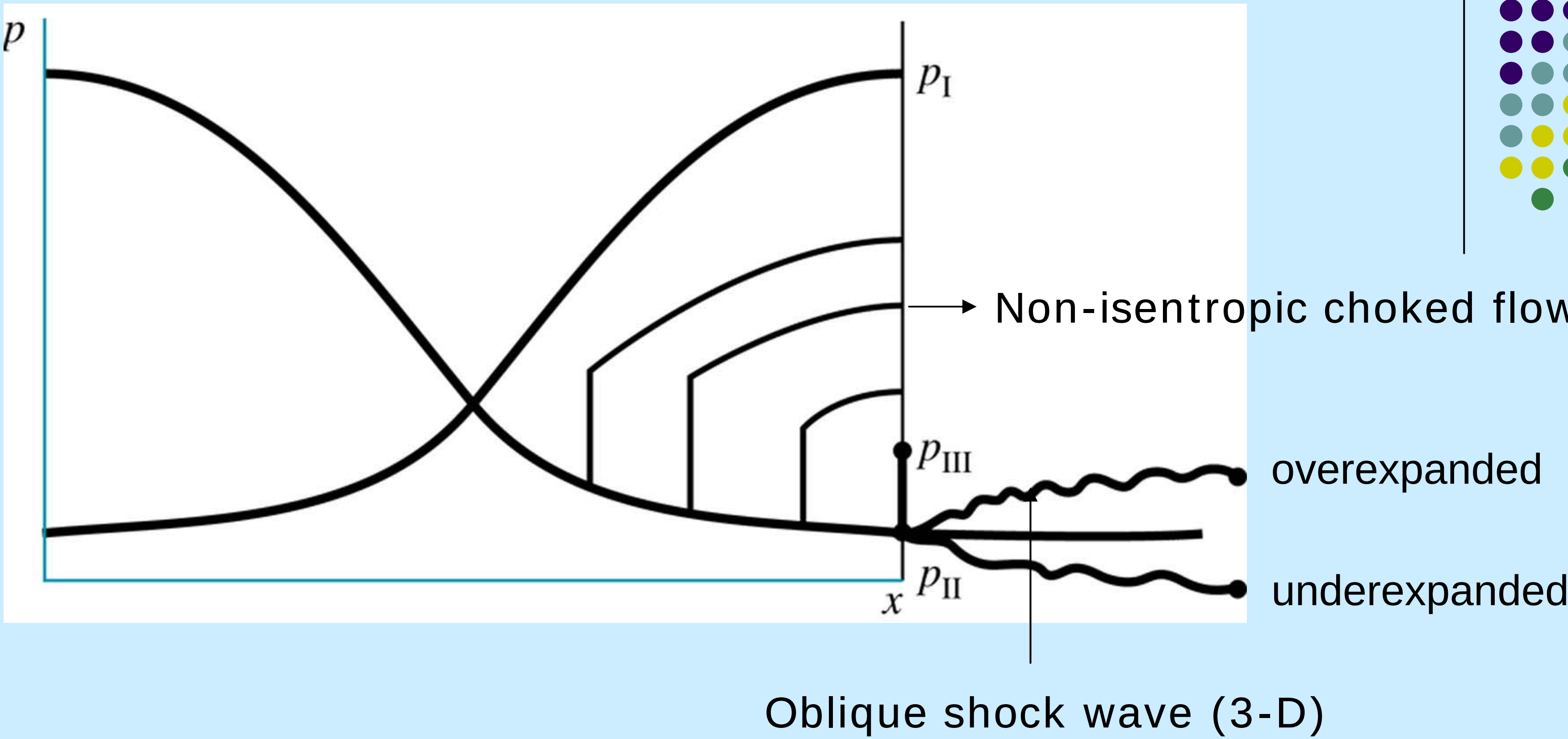
Supersonic-supersonic (not-choked)

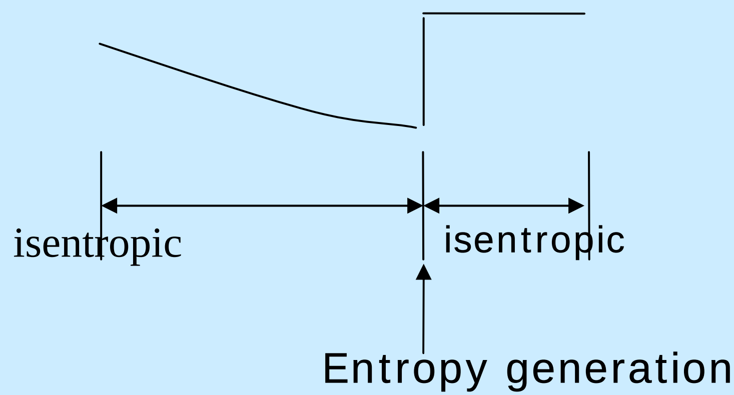
1. For a given (T_0, p_0), k and converging-diverging duct geometry, infinite number of isentropic subsonic to subsonic (not choked) and isentropic supersonic to supersonic (not choked) flow solutions exist.
2. For choked condition, the flow solutions are each unique.



$$p_{ext} \geq p_{\mathrm{I}} \text{ or } p_{ext} \leq p_{\mathrm{II}}, \text{isentropic flow is possible}$$

$$p_{\mathrm{I}} \leq p_{ext} \leq p_{\mathrm{II}} \text{ , isentropic flow is not possible}$$





[V11.5 Rocket engine start-up]

[V11.6 Supersonic nozzle flow]

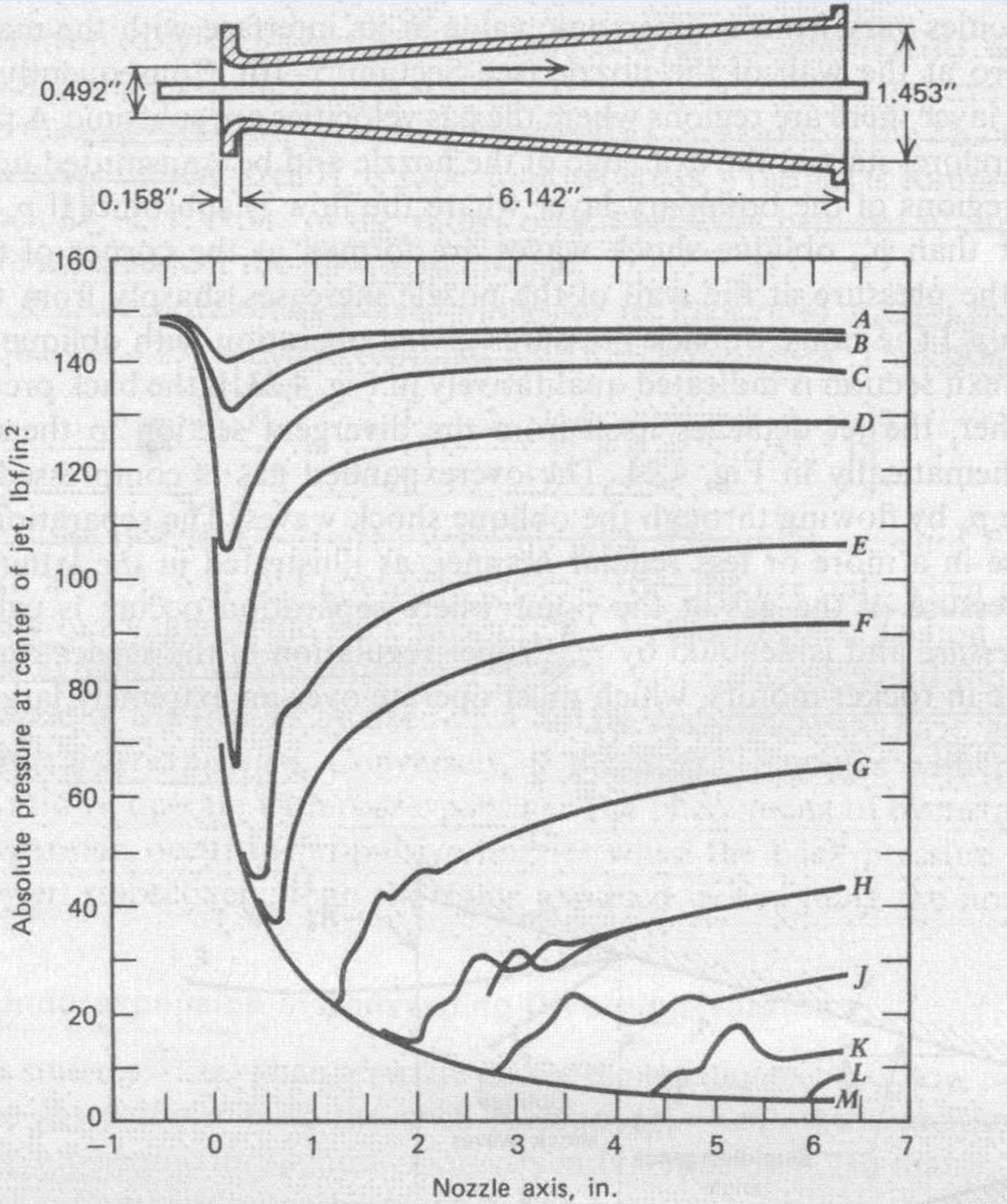


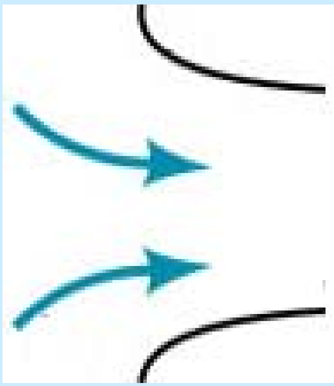
Figure 4.25. Effect of back pressure on the pressure along the axis of a De Laval nozzle (reproduced from Reference 8).

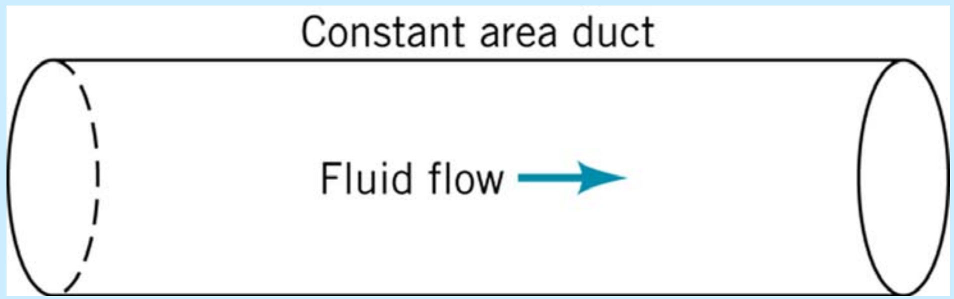
From Zucrow and Hoffman, Gas Dynamics

11.4.3 Constant Area Duct Flow

- For constant area isentropic duct flow, the flow velocity, thus the fluid enthalpy and temperature are constant.

p_0





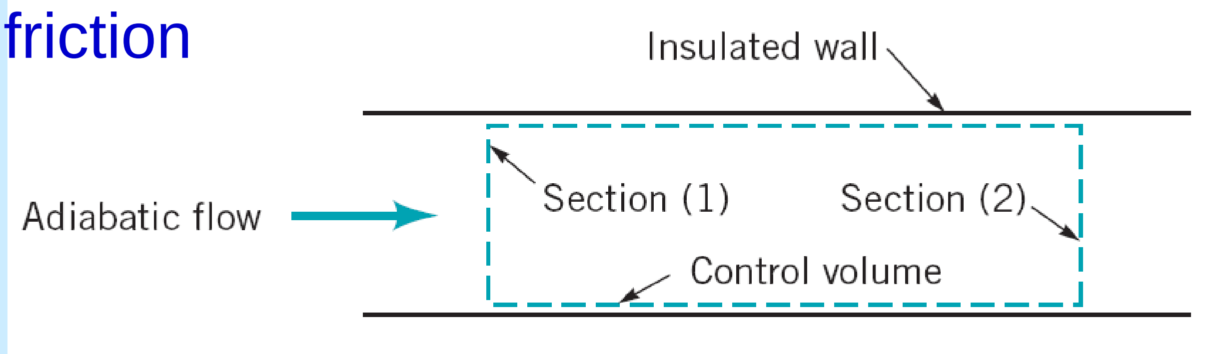
p_b

11.5 Nonisentropic flow of an ideal gas

- **Fanno flow** – adiabatic flow with friction.
- **Rayleigh flow** – constant area duct flow with heat transfer but without friction

11.5.1 Adiabatic constant area duct flow with friction (Fanno flow)

- Consider steady 1-D ideal gas constant area duct flow with friction



Continuity: $\dot{m} = \rho VA = \text{const} \Rightarrow \rho V = \text{const}$

Energy equation:

$$\dot{m}[h_2 - h_1 + \frac{V_2^2 - V_1^2}{2} + g(z_2 - z_1)] = \dot{Q} + \dot{W}$$

$$h + \frac{V^2}{2} = h_0, \quad h - h_0 = c_p(T - T_0)$$

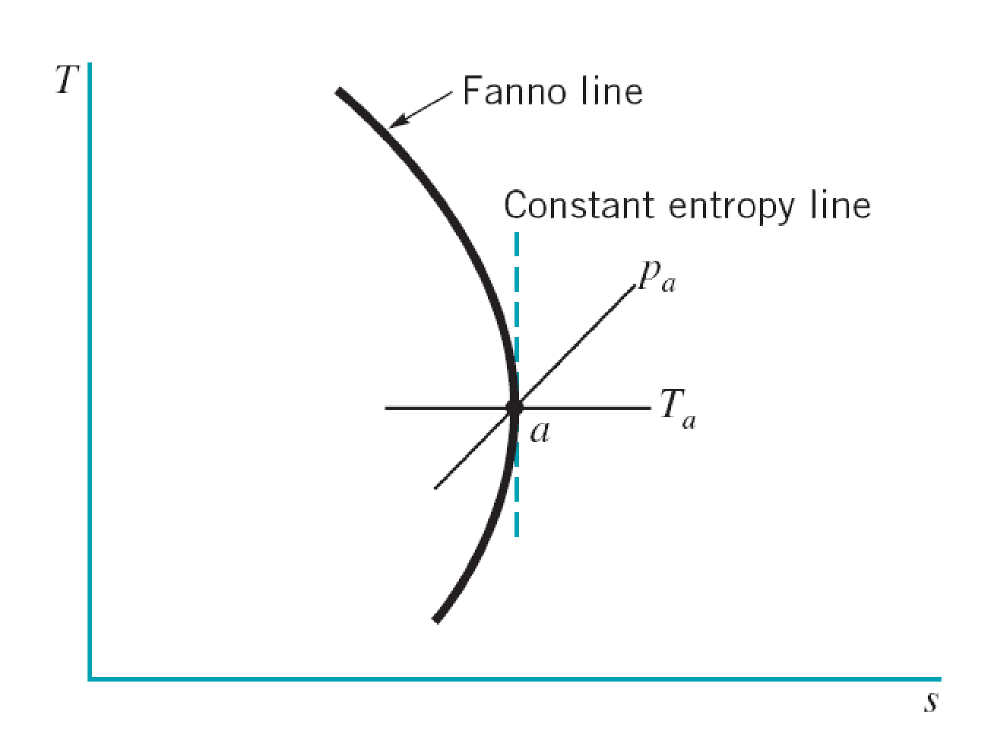
$$T + \frac{V^2}{2c_p} = T_0 = \text{const}$$

(stagnation temp.= const)

$$\Rightarrow \quad T + \frac{(\rho V)^2}{2c_p \rho^2} = T_0 \Rightarrow T + \frac{(\rho V)^2 T^2}{2c_p p^2 / R^2} = T_0, \ \text{where } \rho V = \text{const} \qquad (11.75)$$

Eq. 11.75 allows us to calculate *T* for *p* in the Fanno flow.

Tds equation (2nd law):



$$Tds = dh - (\frac{1}{\rho})dp$$

$$ds = c_p \frac{dT}{T} - R\frac{dp}{p}$$

$$s_2 - s_1 = c_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1}$$

$$s - s_1 = c_p \ln\frac{T}{T_1} - R\ln\frac{p}{p_1} \quad (11.76)$$

$p_1,\ T_1,\ s_1$ are considered reference values from the entrance.

From (11.75) and (11.76), the **Fanno line** for variation of *p*-*T*-*s* can be obtained.

Ex 11.11

Example 11.11 Compressible Flow with Friction (Fanno Flow)

- Air ($k=1.4$) enters [section (1)] an insulated, constant cross-sectional area duct with the following properties:

$T_0=284K$

$T_1=286K$

$p_1=99kPa(abs)$

For Fanno flow, determine corresponding value of fluid temperature and entropy change for various values of downstream pressures and plot the related Fanno line.

Example 11.11

To plot the Fanno line we use Eq. (75) and (76)

$$T + \frac{(\rho V)^2 T^2}{2c_P (p^2 / R^2)} = T_0 = \text{constant} \qquad (11.11.1)$$

$$s - s_1 = c_P \ln \frac{T}{T_1} - R \ln \frac{p}{p_1} \qquad (11.11.2)$$

$$k = 1.4 \qquad R = 286.9 J / kg \cdot K$$

From Eq. (14) $$c_p = \frac{Rk}{k-1} = \ldots = 1004 J / kg \cdot K \qquad (11.11.3)$$

(1)+(69) ⇒ $$\rho V = \frac{p}{RT} Ma \sqrt{RTk} = \rho_1 V_1 = \frac{p_1}{RT_1} Ma_1 \sqrt{RT_1 k} \qquad (11.11.4)$$

Example 11.11

$$\frac{T_1}{T_o} = \frac{286K}{288K} = 0.993$$

$$\frac{\mathrm{T}}{\mathrm{T_o}} = \frac{1}{1 + \frac{\mathrm{k}-1}{2}\mathrm{Ma}^2} \quad (56)$$

From Eq. (56) $\mathrm{M}a_1 = \sqrt{\left(\frac{1}{0.993} - 1\right) / 0.02} = 0.2$

$$\sqrt{RT_1k} = \ldots = 339m/s$$

(11.11.4) $$\rho \mathrm{V} = \frac{99 \times 10^3 Pa\, 0.2(339m/s)}{(289.6J/kg \cdot K)(286K)} = 81.8kg/(m^2 \cdot s)$$

For p= 48 kPa

(11.11.1) $$T + \frac{(\rho V)^2 T^2}{2c_P(p^2/R^2)} = \ldots = 288K \Rightarrow \mathrm{T} = 278.7K$$

(11.11.2) $$s - s_1 = c_P \ln\frac{T}{T_1} - R\ln\frac{p}{p_1} = \ldots = 181.7J/(kg \cdot K)$$

Example 11.11

For p=48kPa T=278.7K $s-s_1$=181.7J/(kg・K)

For p=41kPa T=275.6K $s-s_1$=215.7J/(kg・K)

For p=34kPa T=270.6K $s-s_1$=251.0J/(kg・K)

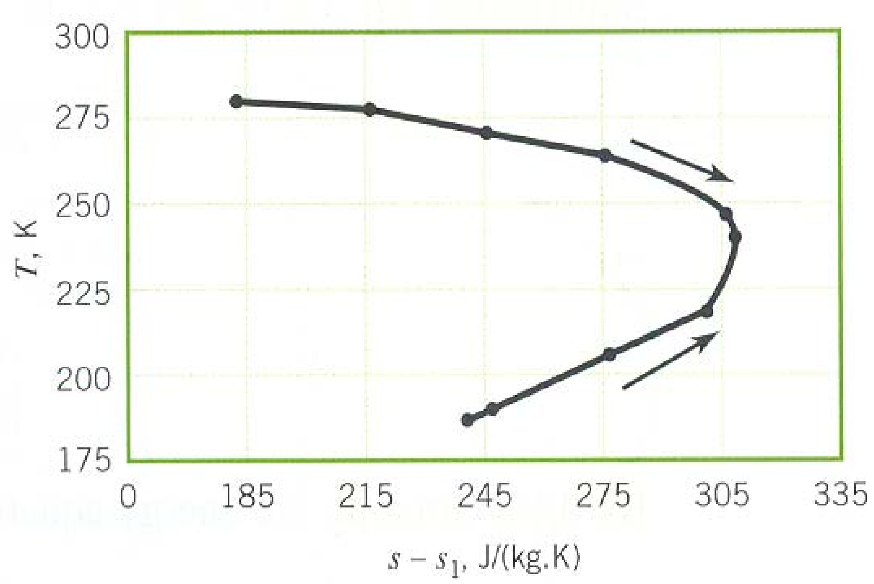


FIGURE E11.11

Analysis of Fanno line

Tds equation

$$Tds = dh - \frac{dp}{\rho} = dh - RT\frac{dp}{p}$$

For an ideal gas,

$$Tds = c_P dT - RT\frac{dp}{p} \qquad \because dh = c_P dT;$$

$$= c_P dT - RT(\frac{d\rho}{\rho} + \frac{dT}{T}) \qquad p = \rho RT \quad \text{or} \quad \frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

Continuity: $\rho V = \text{const}, \text{ or } \frac{d\rho}{\rho} = -\frac{dV}{V}$

$$\rightarrow Tds = c_P dT - RT(\frac{d\rho}{\rho} + \frac{dT}{T}) = c_P dT - RT(-\frac{dV}{V} + \frac{dT}{T})$$

$$\rightarrow \frac{ds}{dT} = \frac{c_P}{T} - R(-\frac{1}{V}\frac{dV}{dT} + \frac{1}{T})$$

Energy eq.: $T + \frac{V^2}{2c_p} = T_0 = \text{const} \rightarrow dT = -\frac{VdV}{c_p} \Rightarrow \frac{dV}{dT} = -\frac{c_P}{V}$

$$\therefore \frac{ds}{dT} = \frac{c_p}{T} - R(\frac{c_p}{V^2} + \frac{1}{T}) \qquad (11.82)$$

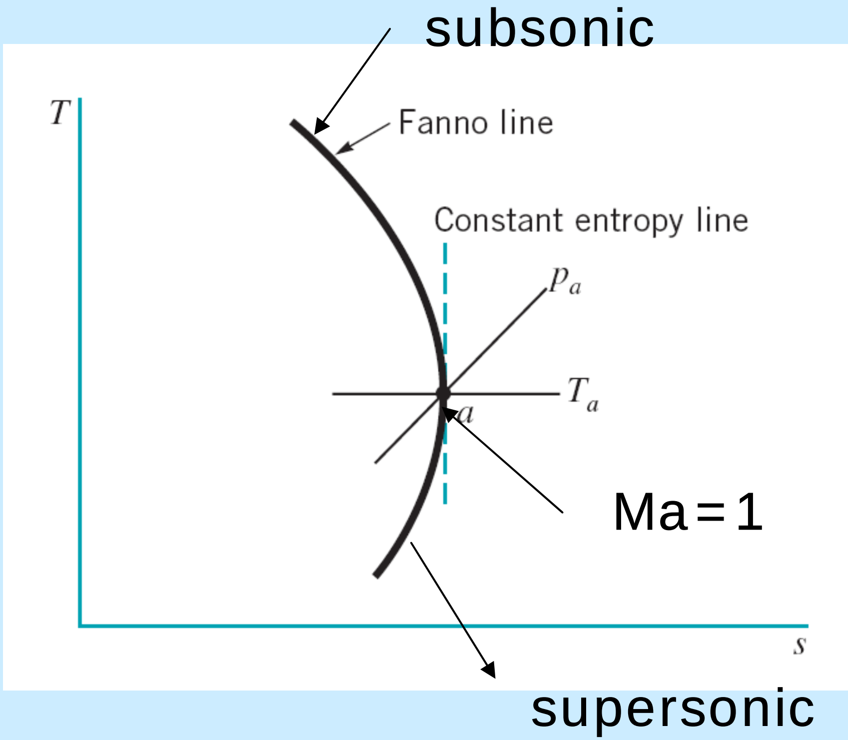
$$\text{For } \frac{ds}{dT} = 0 \rightarrow \frac{c_p}{T} = R\left(\frac{c_p}{V^2} + \frac{1}{T}\right) \rightarrow c_p - R = c_\upsilon = RT\frac{c_p}{V^2}$$

$$V = \sqrt{(c_p / c_\upsilon)RT_a} = \sqrt{kRT_a}$$

So, the Mach number at state *a* is 1.

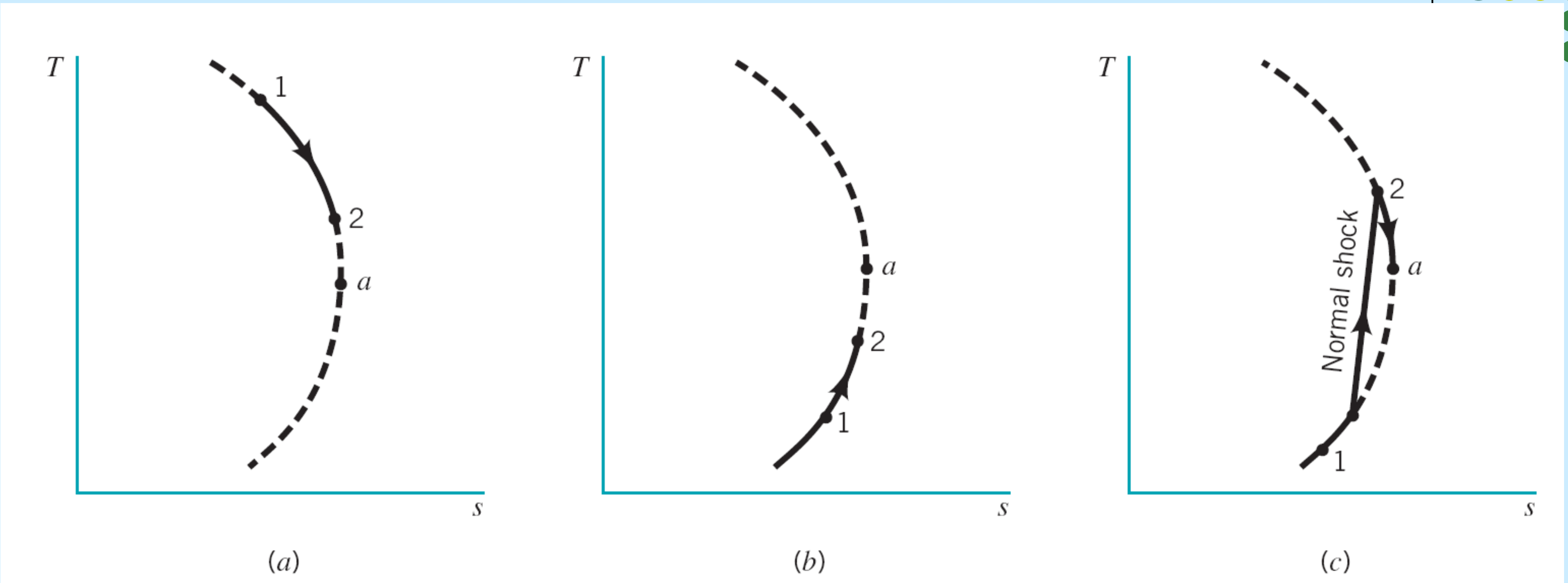
$$\frac{ds}{dT} < 0,\ V < \sqrt{kRT_a} \rightarrow \text{subsonic}$$

$$\frac{ds}{dT} > 0,\ V > \sqrt{kRT_a} \rightarrow \text{supersonic}$$



Since T_0 is constant on the Fanno line, the temperature at point *a* is the critical temperature T^*.

-2nd law $ds>0$



subsonic flow (acceleration)

supersonic flow (deceleration)

Normal shock

Summary of Fanno flow behavior

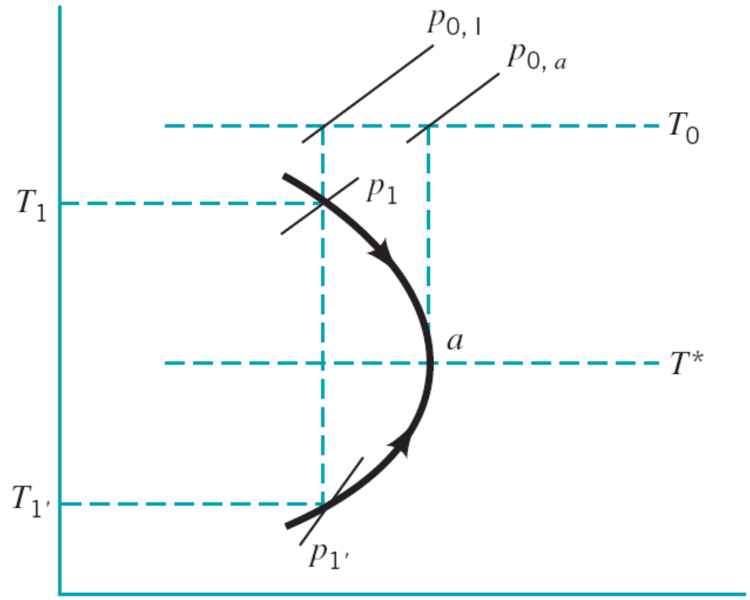


TABLE 11.1

Summary of Fanno Flow Behavior

	Flow	
Parameter	**Subsonic Flow**	**Supersonic Flow**
Stagnation temperature	Constant	Constant
Ma	Increases (maximum is 1)	Decreases (minimum is 1)
~~Friction~~	~~Accelerates flow~~	~~Decelerates flow~~
Pressure	Decreases	Increases
Temperature	Decreases	Increases

Friction drags acceleration by pressure drop

Friction helps deceleration by pressure rise

-To quantify the Fanno flow behavior, we need to combine relationship that represents the linear momentum law with the set of equations already derived.

$$p_1A_1 - p_2A_2 - R_x = m(V_2 - V_1)$$

$$\rightarrow p_1 - p_2 - \frac{R_x}{A} = \rho V(V_2 - V_1), \quad (\because A_1 = A_2 = A \text{ and } \dot{m} = \rho AV = C)$$

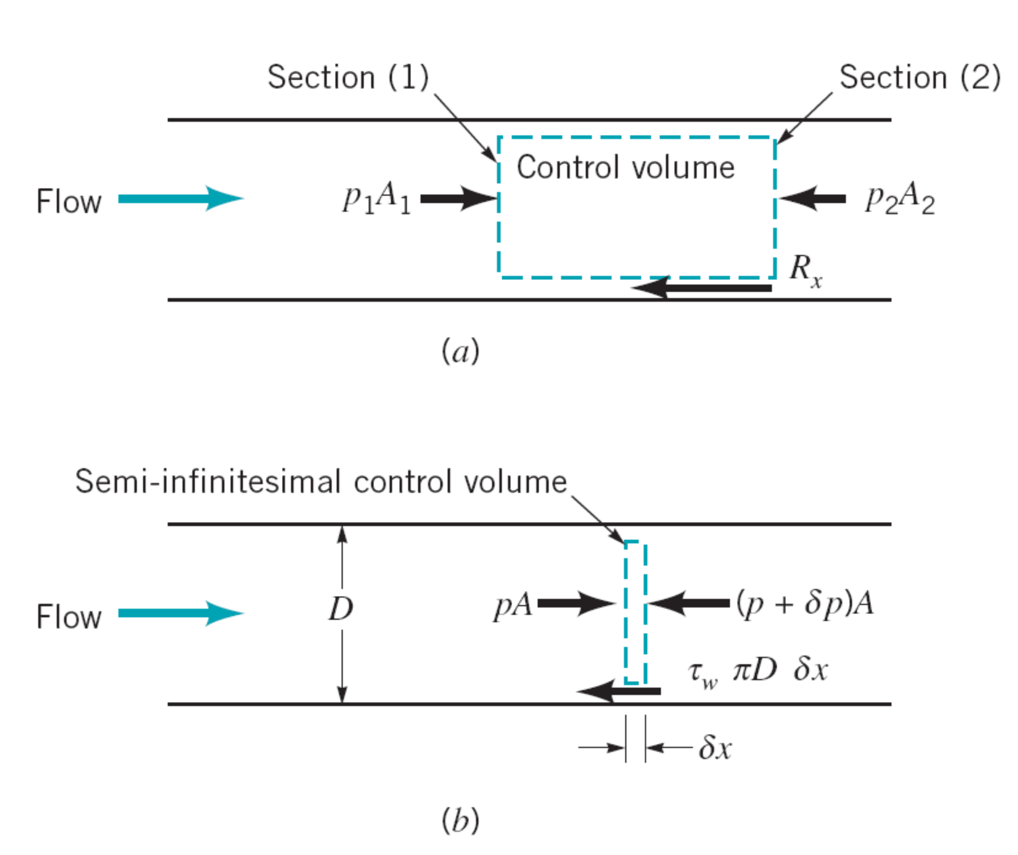
-Therefore, for the semi-infinitesimal control volumes

$$-dp - \frac{\tau_w \pi D dx}{A} = \rho V dV$$

with $f = \frac{8\tau_w}{\rho V^2}, \ A = \frac{\pi D^2}{4}$

$$\rightarrow \ -dp - f\rho\frac{V^2 dx}{2D} = \rho V dV$$

or $$\frac{dp}{p} + \frac{f}{p}\frac{\rho V^2}{2}\frac{dx}{D} + \frac{\rho}{p}\frac{d(V^2)}{2} = 0$$



$$\frac{dp}{p}+\frac{f}{p}\frac{\rho V^2}{2}\frac{dx}{D}+\frac{\rho}{p}\frac{d(V^2)}{2}=0 \qquad (11.88)$$

After some derivation, (11.88) becomes

$$\frac{1}{2}(1+k\mathrm{Ma}^2)\frac{d(V^2)}{V^2}-\frac{d(\mathrm{Ma}^2)}{\mathrm{Ma}^2}+\frac{fk}{2}\mathrm{Ma}^2\frac{dx}{D}=0$$

or $$\frac{(1-\mathrm{Ma}^2)d(\mathrm{Ma}^2)}{\left\{1+[(k-1)/2]\mathrm{Ma}^2\right\}k\mathrm{Ma}^4}=f\frac{dx}{D} \qquad (11.96)$$

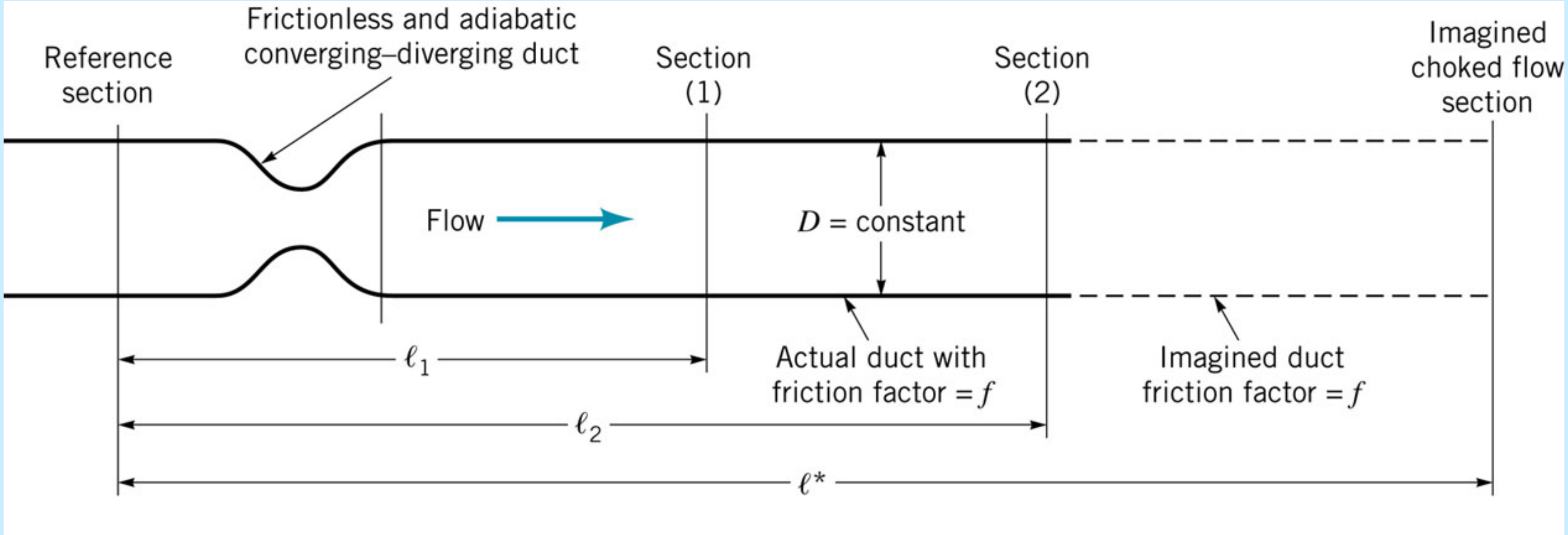
-Integrate to the critical state

$$\int_{\mathrm{Ma}}^{\mathrm{Ma}^*=1}\frac{(1-\mathrm{Ma}^2)d(\mathrm{Ma}^2)}{\left[1+[(k-1)/2]\mathrm{Ma}^2\right]k\mathrm{Ma}^4}=\int_{\ell}^{\ell^*}f\frac{dx}{D}$$

$$\frac{1}{k}\frac{\left(1-\mathrm{Ma}^2\right)}{\mathrm{Ma}^2}+\frac{k+1}{2k}\ell n\left[\frac{[(k+1)/2]\mathrm{Ma}^2}{1+[(k-1)/2]\mathrm{Ma}^2}\right]=\frac{f\left(\ell^*-\ell\right)}{D} \qquad (11.98)$$

-Note that the critical state does not have to exist, since for any two section in the Fanno flow,

$$\frac{f\left(\ell^{*}-\ell_{2}\right)}{D}-\frac{f\left(\ell^{*}-\ell_{1}\right)}{D}=\frac{f}{D}\left(\ell_{1}-\ell_{2}\right).$$



(a)

-Other fluid properties in the Fanno flow can also be derived, as summarized below.

Summary of Property Relations for Fanno Flow

Note: These equations correlate ratio of properties associated with different positions (a certain position and the choke position), between which friction loss exists.

$$\frac{1}{k}\frac{\left(1-\mathrm{Ma}^2\right)}{\mathrm{Ma}^2}+\frac{k+1}{2k}\ell n\left[\frac{[(k+1)/2]\mathrm{Ma}^2}{1+[(k-1)/2]\mathrm{Ma}^2}\right]=\frac{f\left(\ell^*-\ell\right)}{D} \quad (11.98)$$

$$\frac{T}{T^*}=\frac{(k+1)/2}{1+[(k-1)/2]\mathrm{Ma}^2}, \quad (11.101)$$

$$\frac{V}{V^*}=\left[\frac{[(k+1)/2]\mathrm{Ma}^2}{1+[(k-1)/2]\mathrm{Ma}^2}\right]^{1/2}, \quad (11.103)$$

Note: For p/p_0, p^*/p_0^*, T/T_0 at the same position, isentropic relations in terms of Ma or Fig. D1 can be used.

$$\frac{\rho}{\rho^*}=\left(\frac{V}{V^*}\right)^{-1}, \quad (11.105)$$

$$\frac{p}{p^*}=\frac{\rho}{\rho^*}\frac{T}{T^*}=\frac{1}{\mathrm{Ma}}\left[\frac{(k+1)/2}{1+[(k-1)/2]\mathrm{Ma}^2}\right]^{1/2}, \quad (11.107)$$

same position — different positions

$$\frac{p_0}{p_0{}^*}=\frac{p_0}{p}\frac{p}{p^*}\frac{p^*}{p_0{}^*}=\frac{1}{\mathrm{Ma}}\left[\left(\frac{2}{k+1}\right)\left(1+[(k-1)/2]\mathrm{Ma}^2\right)\right]^{\frac{k+1}{2(k-1)}} \quad (11.109)$$

Note: These curves correlate ratio of properties associated with different positions (a certain position and the choke position), between which friction loss exists.

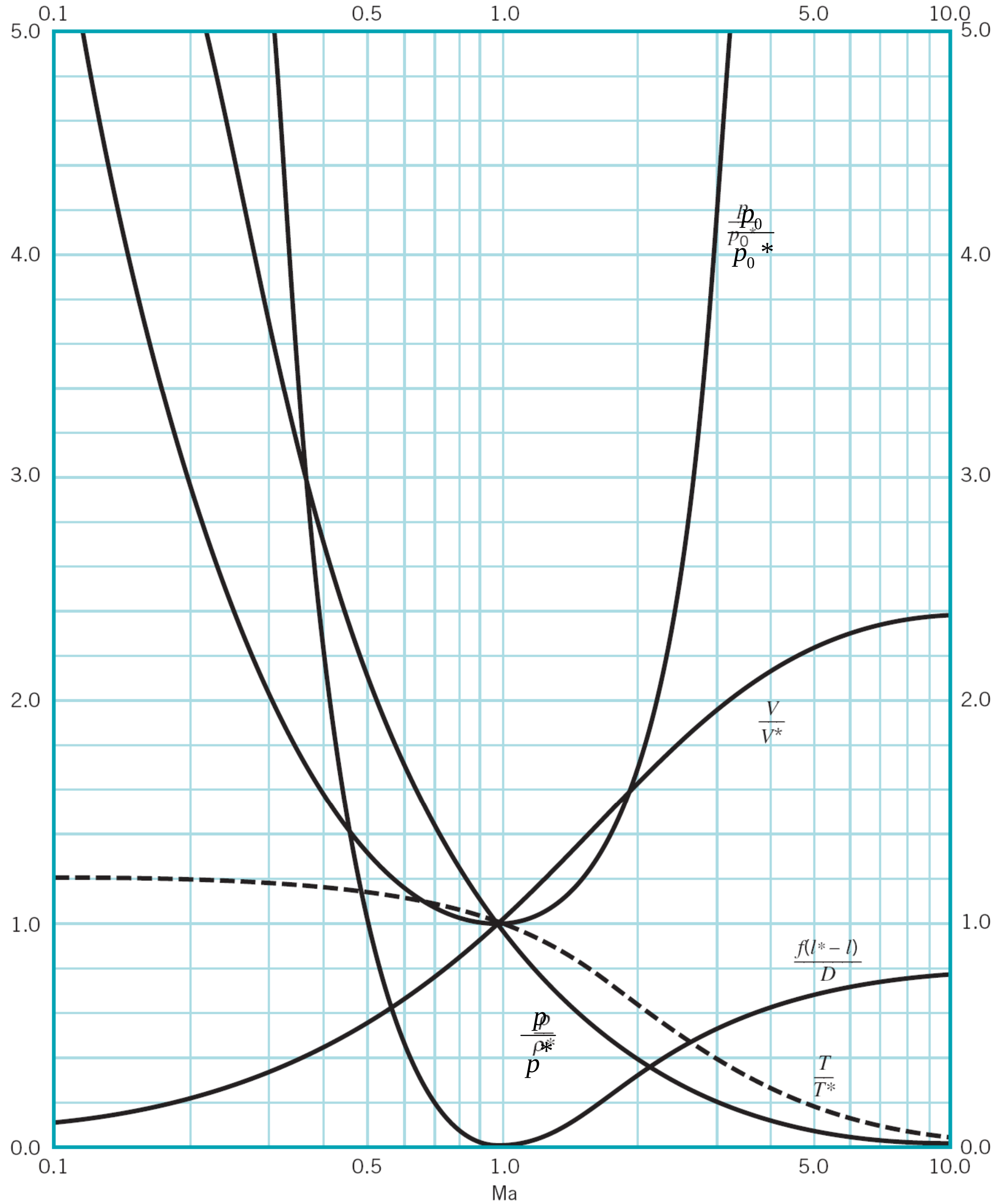


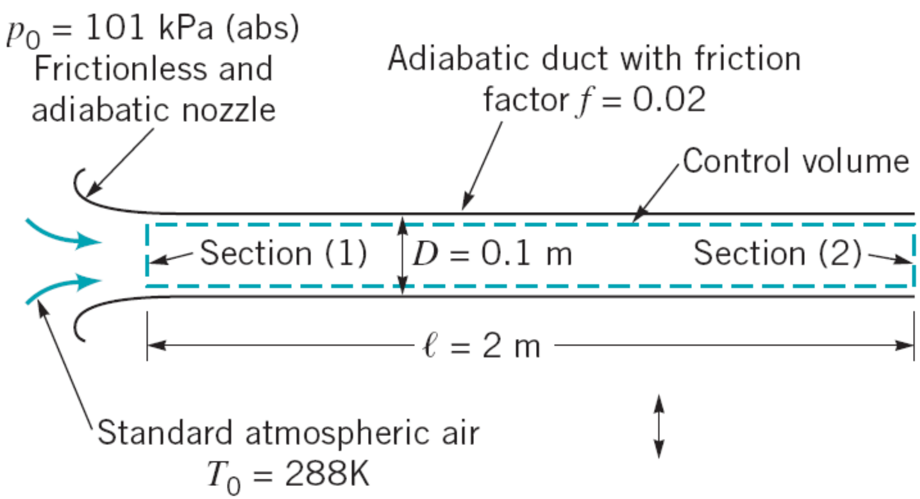
Figure D2 (p. 719)

Fanno flow of an ideal gas with ***k*** **= 1.4.**

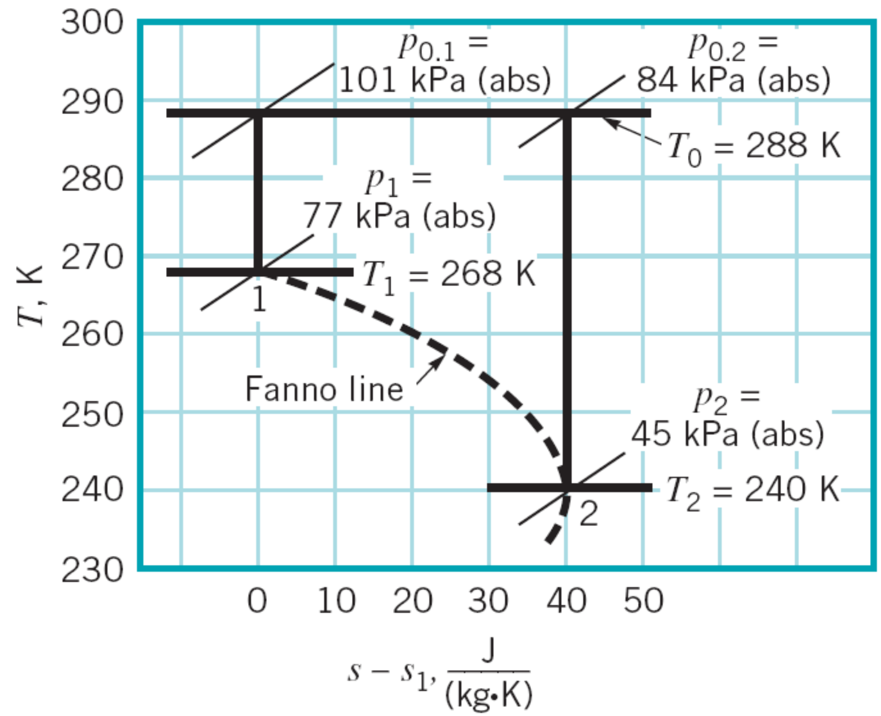
Example 11.12 Choked Fanno flow

Given $p_0 = 101\text{ kPa},\ T_0 = 288\text{ K}$

What is the maximum flow rate at the duct?



(a)



(b)

-For maximum flow rate, the flow must be choked at the exit.

$$\frac{f\left(\ell^* - \ell_1\right)}{D} = \frac{f\left(\ell_2 - \ell_1\right)}{D} = \frac{0.02 \times 2}{0.1} = 0.4$$

$$\text{Fig. D2} \rightarrow \text{Ma}_1 = 0.63 \rightarrow \frac{T_1}{T^*} = 1.1, \ \frac{V_1}{V^*} = 0.66, \ \frac{p_1}{p^*} = 1.7, \ \frac{p_{0,1}}{p_0^*} = 1.16$$

$$\text{Fig. D1 with Ma}_1 = 0.63 \rightarrow \frac{T_1}{T_0} = 0.93, \ \frac{p_1}{p_{0,1}} = 0.76, \ \frac{\rho_1}{\rho_{0,1}} = 0.83$$

Since T_0=C=288 K

$$\frac{T^*}{T_0} = \frac{2}{k+1} = 0.8333 \Rightarrow T^* = T_0(0.8333) = 240 \Leftarrow T_2$$

$$V^* = \sqrt{RT^*k} = 310\text{m/s} \ (\Leftarrow V_2) \rightarrow V_1 = 0.66\ V^* = 205\text{m/s}$$

$$\rho_{0,1} = p_{0,1} / RT_{0,1} = 1.23 \rightarrow \rho_1 = 0.83\rho_{0,1} = 1.02 \text{ kg/m}^3$$

$$\dot{m} = \rho_1 V_1 A = 1.65 \text{ kg/s}$$

Or, perhaps more logical,

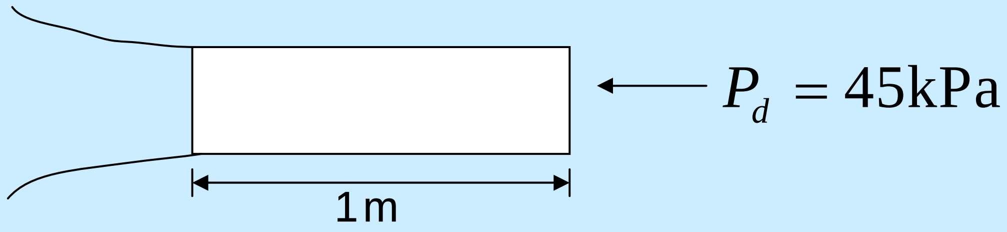
$$T_1 = 0.93 \times T_0 = 0.93 \times 288$$
$$= 268 \text{ K}$$
$$V_1 = \text{Ma}_1 \times \sqrt{kRT_1}$$
$$= 0.63 \times \sqrt{1.4 \times 386.9 \times 268}$$
$$= 207\text{m/s}$$

with friction, use Fig. D2

$$p_2 = \frac{p^*}{p_1} \frac{p_1}{p_{0,1}} p_{0,1} = \frac{1}{1.7} \times 0.76 \times 101 = 45\text{kPa}$$

$$p_0^* = p_{0,2} = p_{0,1} \times \frac{1}{1.16} = 87\text{kPa}$$

Example 11.13 Effect of duct length on choked Fanno flow



If the flow is choked.

$$\frac{f\left(\ell^* - \ell_1\right)}{D} = \frac{0.02\times 1}{0.1} = 0.2, \quad \text{Fig. D2} \rightarrow \text{Ma}_1 = 0.7, \ \frac{p_1}{p^*} = 1.5, \ \frac{V_1}{V^*} = 0.73,$$

$$\text{Fig. D1 with Ma}_1 = 0.70 \rightarrow \ \frac{p_1}{p_0} = 0.72, \ \frac{\rho_1}{\rho_{0,1}} = 0.79$$

$$p_2 = p^* = \frac{p^*}{p_1}\frac{p_1}{p_{01}} p_{01} = \frac{1}{1.5}\times 0.72\times 101 = 48.5\text{kPa} \ (> p_d = 45\text{kPa})$$

$$\rho_1 = 0.79\rho_{0,1}, \ V_1 = 0.73V^*$$

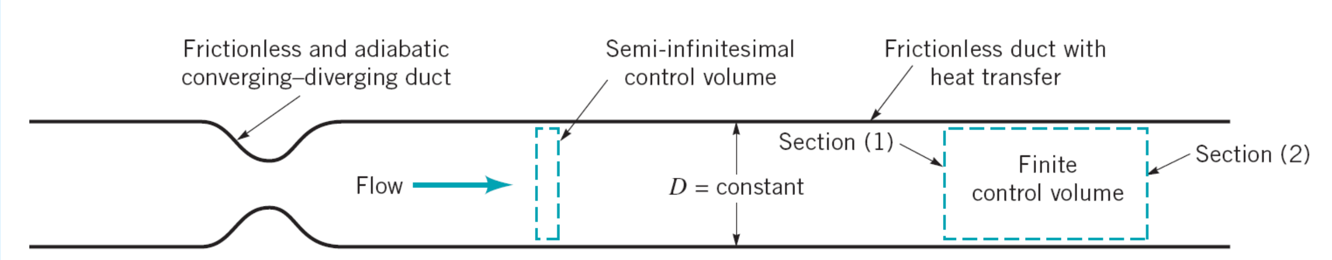
$$\rightarrow \dot{m} = \rho_1 A_1 V_1 = 1.73 \text{ kg/s}$$

-For the same upstream stagnation state and downstream pressure,

$$\ell\uparrow \rightarrow \dot{m}\downarrow \quad \text{or} \quad \text{when } \ell \text{ is fixed: } f\uparrow \rightarrow \dot{m}\downarrow$$

Example 11.14 Unchoked Fanno flow

11.5.2 Frictionless constant-area duct flow with heat transfer (Rayleigh flow)



Momentum: $p_1A_1 + mV_1 = p_2A_2 + mV_2 + \not{R}_x$ → 0 (frictionless flow)

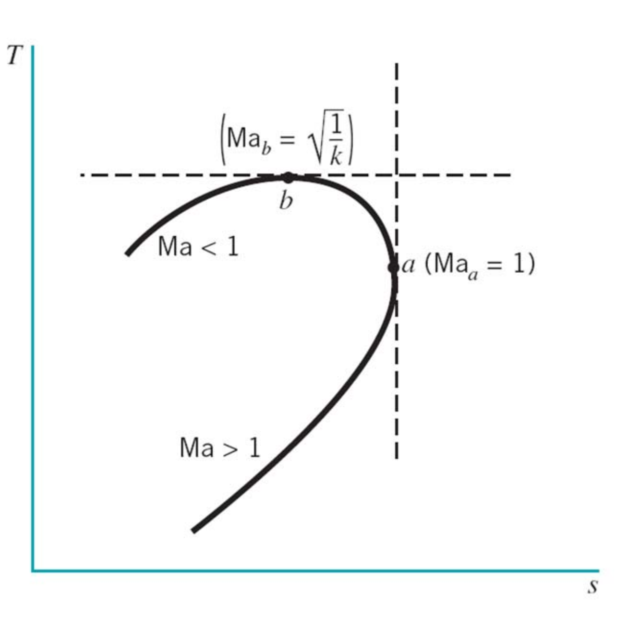
or $$p + \frac{(\rho V)^2}{\rho} = \text{const} \;\rightarrow\; p + \frac{(\rho V)^2 RT}{p} = \text{const}$$ ——(11.111)

$p_0 = p + \dfrac{\rho V^2}{2} \neq$ const, although there is no friction, why?

Continuity: $\rho V = C$

Tds eq.: $$s - s_1 = c_p \ell n \frac{T}{T_1} - R \ell n \frac{p}{p_1}$$ ——(11.76)

Eqs. (11.111) and (11.76) can be used to construct a Rayleigh line with reference conditions.



Example 11.15 Construction of a Rayleigh line for various downstream p_2 (or T_2), with entrance conditions T_0, T_1, p_1

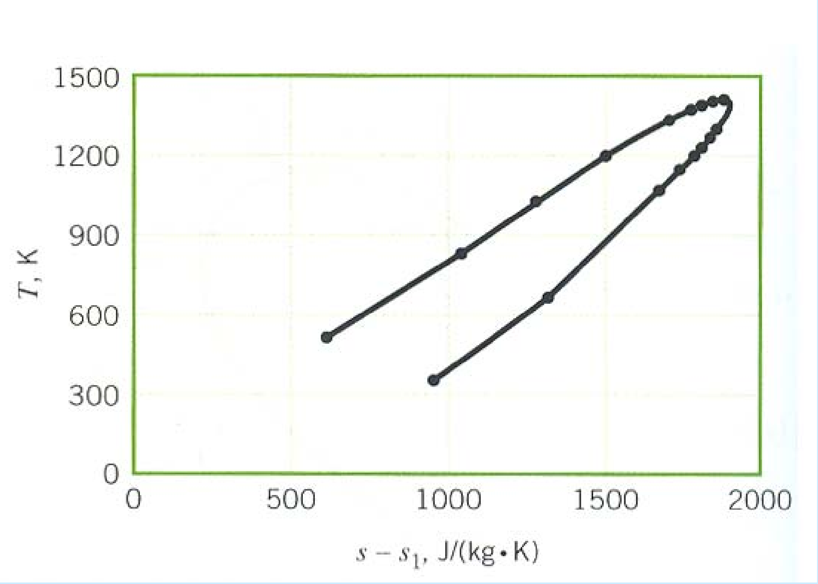
Given $T_0,\ T_1, p_1, \rightarrow \rho_1,\ V_1$ and $\rho V = C$ can be calculated.

-Assume p_2 (or T_2), then from (11.111) T_2 (or p_2) can be obtained.

-From (11.76), s_2 can be obtained.

$$p + \frac{(\rho V)^2 RT}{p} = \text{const} \qquad (11.111)$$

$$s - s_1 = C_p \ell n \frac{T}{T_1} - R \ell n \frac{p}{p_1} \qquad (11.76)$$



p kPa (abs)	T (K)	$s - s_1$ [J/(kg · K)]
93	534	645
86	807	1082
79	1028	1349
72	1199	1530
62	1356	1697
55	1404	1766
52	1409	1786
51.5	1409	1789
48	1400	1802
43	1366	1809
41	1346	1808
38	1306	1800
34	1240	1799
31	1179	1755
28	1109	1723
14	656	1395
7	354	974

Discussion on the Rayleigh line

-At point *a* on the Rayleigh line, $\dfrac{ds}{dT}=0$

-After some derivation, we have

$$\frac{ds}{dT}=\frac{c_p}{T}+\frac{V}{T}\frac{1}{\left[(T/V)-(V/R)\right]} \qquad (11.115)$$

For $\dfrac{ds}{dT}=0,$

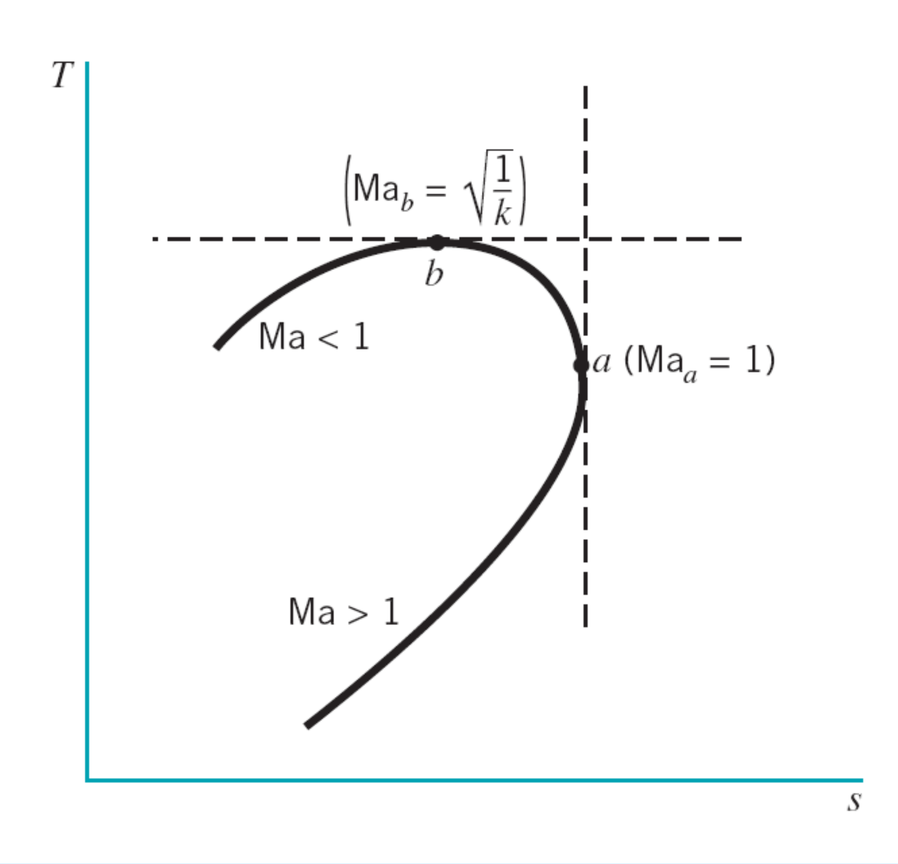
$$\frac{ds}{dT}=\frac{c_p}{T}+\frac{V}{T}\frac{1}{\left[(T/V)-(V/R)\right]}$$

$$\frac{kR}{k-1}\left(\frac{T}{V}-\frac{V}{R}\right)+V=0$$

$$\frac{kRT}{k-1}-\frac{kV^2}{k-1}+\frac{(k-1)V^2}{k-1}=0$$

$$V^2=kRT$$

$$V_a=\sqrt{kRT_a} \quad \Rightarrow \quad \mathrm{Ma}_a=1$$



Derivation of *ds/dT*

$$dp = -\rho V dV$$

$$\frac{dP}{\rho} = -VdV$$

Since *Tds* equation

$$Tds = dh - \frac{dp}{\rho}$$

$$= c_p dT + VdV$$

or

$$\frac{ds}{dT} = \frac{c_p}{T} + \frac{V}{T}\frac{dV}{dT}$$

$$= \frac{c_p}{T} + \frac{V}{T}\frac{1}{\left(T/V - V/R\right)}$$

$$\left[\begin{array}{l} \rho V = C \\ \dfrac{d\rho}{\rho} = -\dfrac{dV}{V} \end{array}\right.$$

$$p = \rho RT$$

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{.dT}{T}$$

$$-\frac{\rho V dV}{\rho RT} = -\frac{dV}{V} + \frac{dT}{T}$$

$$-\frac{V}{RT} = -\frac{1}{V} + \frac{1}{T}\frac{dT}{dV}$$

$$\frac{T}{V} - \frac{V}{R} = \frac{dT}{dV}$$

At point b, $dT/ds = 0$

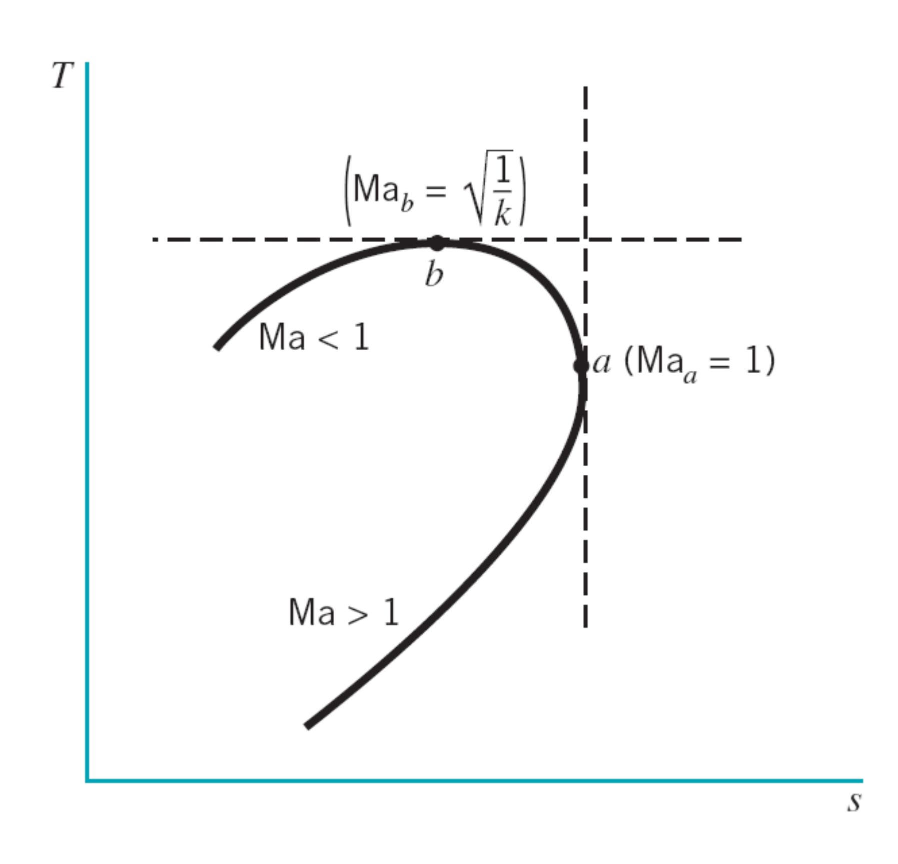
$$\frac{ds}{dT} = \frac{c_p}{T} + \frac{V}{T}\frac{1}{\left[(T/V) - (V/R)\right]}$$

$$\frac{dT}{ds} = \frac{1}{\dfrac{ds}{dT}} = \frac{1}{\dfrac{c_p}{T} + \dfrac{V}{T}\left[(T/V) - (V/R)\right]^{-1}}$$

$$\frac{c_p}{T} + \frac{V}{T}\frac{1}{\left[(T/V) - (V/R)\right]} \rightarrow \infty$$

$$\therefore \frac{T}{V} = \frac{V}{R} \Rightarrow V^2 = RT \Rightarrow V = \sqrt{RT}$$

$$\mathrm{Ma}_b = \frac{V_b}{c} = \frac{\sqrt{RT}}{\sqrt{kRT}} = \sqrt{\frac{1}{k}}$$



- At point b, the flow is subsonic since $k > 1$.

- Now, consider the energy equation,

$$\dot{m}\left[h_2 - h_1 + \frac{V_2^{\ 2} - V_1^{\ 2}}{2} + g\left(z_2 - z_1\right)\right] = Q_{net} + W_{sheftnet}$$

$$dh + VdV = \delta q, \quad dh = c_p dT = \frac{kR}{k-1}dT$$

$$c_p dT + VdV = \delta q$$

$$\frac{dT}{T} + \frac{VdV}{c_p T} = \frac{\delta q}{c_p T}$$

$$\frac{dV}{V}\left[\frac{V}{T}\frac{dT}{dV} + \frac{V^2}{kRT/\left(k-1\right)}\right] = \frac{\delta q}{c_p T}$$

$$\therefore \frac{dV}{V} = \frac{\delta q}{c_p T}\left[\frac{V}{T}\frac{dT}{dV} + \frac{\left(k-1\right)V^2}{kRT}\right]^{-1} = \frac{\delta q}{c_p T}\left[\frac{V}{T}\left(\frac{T}{V} - \frac{V}{R}\right) + \left(k-1\right)\mathrm{Ma}^2\right]^{-1}$$

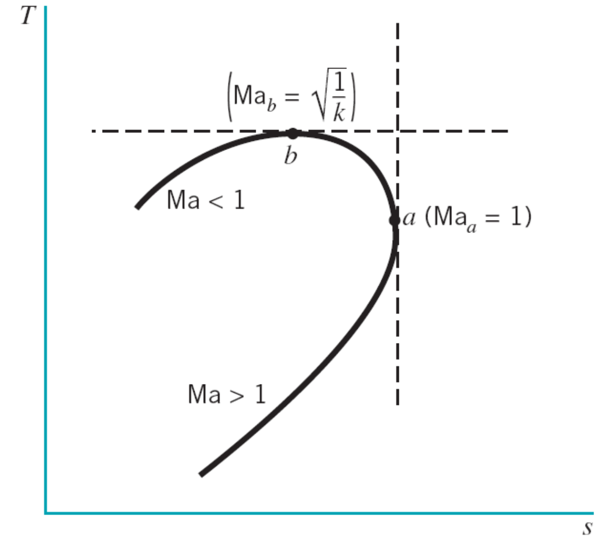
$$= \frac{\delta q}{c_p T}\left[1 - \frac{V^2}{RT} + \left(k-1\right)\mathrm{Ma}^2\right]^{-1} = \frac{\delta q}{c_p T}\left[1 - k\mathrm{Ma}^2 + \left(k-1\right)\mathrm{Ma}^2\right]^{-1}$$

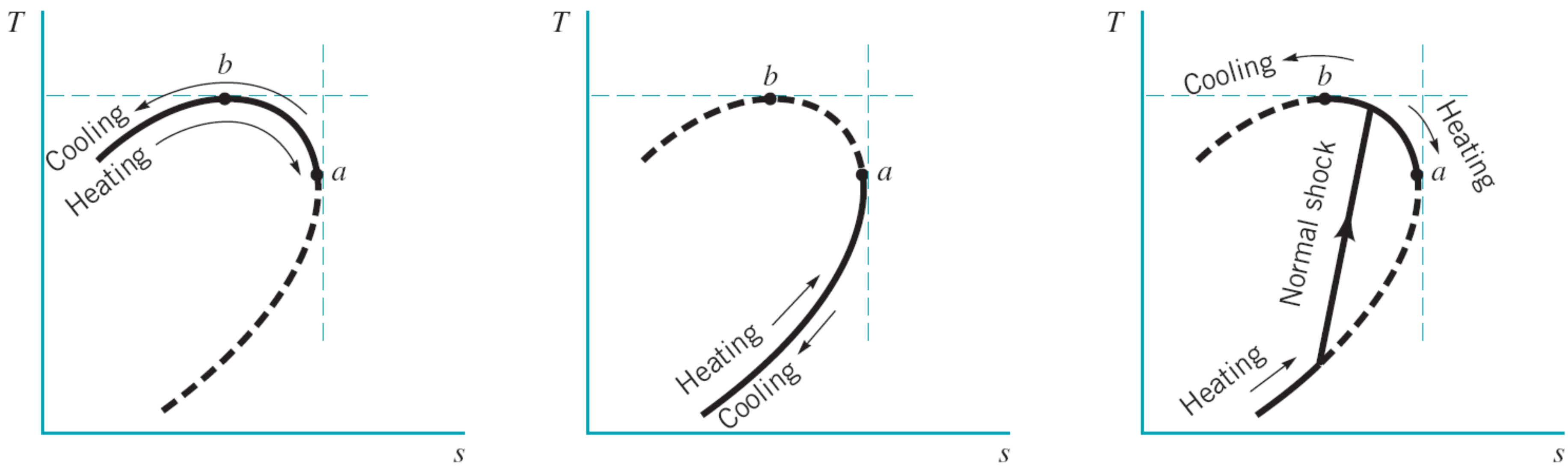
$$= \frac{\delta q}{c_p T}\frac{1}{\left[1 - \mathrm{Ma}^2\right]} \qquad (11.121)$$

TABLE 11.2

Summary of Rayleigh Flow Characteristics

	Heating	Cooling
Ma < 1	Acceleration	Deceleration
Ma > 1	Deceleration	Acceleration





(a) (b) (c)

Derivation for Property Relations for Rayleigh Flow

-linear momentum

$$p + \rho V^2 = p_a + \rho_a V_a^{\,2}$$ where *a* is the reference state.

or $$\frac{p}{p_a} + \frac{\rho V^2}{p_a} = 1 + \boxed{\frac{\rho_a}{p_a} V_a^{\,2}}$$

$$\frac{\rho_a}{p_a} V_a^{\,2} = \frac{\rho_a}{\rho_a R T_a} V_a^{\,2} = \frac{k V_a^{\,2}}{k R T_a} = k, \text{ since } V_a = \sqrt{kRT_a}$$

$$\frac{p}{p_a} + \frac{\rho V^2}{p_a} = 1 + \frac{\rho_a}{p_a} V_a^{\,2} = 1 + k$$

$$\frac{p}{p_a}\left(1 + \frac{\rho V^2}{p}\right) = 1 + k, \quad \text{where } V = \mathrm{Ma}\sqrt{kRT}$$

$$\frac{p}{p_a} = \frac{1+k}{1+k\mathrm{Ma}^2} \qquad (11.123)$$

$$\frac{\rho_a}{\rho} = \frac{V}{V_a} = \mathrm{Ma}\sqrt{\frac{T}{T_a}} \Rightarrow \frac{T}{T_a} = \frac{p}{p_a}\mathrm{Ma}\sqrt{\frac{T}{T_a}}$$

$$\frac{T}{T_a} = \frac{p}{p_a}\frac{\rho_a}{\rho} \rightarrow \frac{T}{T_a} = \left(\frac{p}{p_a}\mathrm{Ma}\right)^2 = \left[\frac{(1+k)\mathrm{Ma}}{1+k\mathrm{Ma}^2}\right]^2 \qquad \left(\because \frac{p}{p_a} = \frac{1+k}{1+k\mathrm{Ma}^2}\right)$$

$$\frac{\rho_a}{\rho} = \frac{V}{V_a} = \mathrm{Ma}\sqrt{\frac{T}{T_a}} = \mathrm{Ma}\left[\frac{(1+k)\mathrm{Ma}}{1+k\mathrm{Ma}^2}\right] \qquad (11.129)$$

- Due to heat transfer, T_0 varies

$$\frac{T_0}{T_{0,a}} = \frac{T_0}{T}\frac{T}{T_a}\frac{T_a}{T_{0,a}}$$

$$= \left[1+[(k-1)/2]\mathrm{Ma}^2\right]\left[\frac{(1+k)\mathrm{Ma}}{1+k\mathrm{Ma}^2}\right]\left[\frac{1}{1+(k-1)/2}\right]$$

$$= \frac{2(k+1)\mathrm{Ma}^2\left(1+[(k-1)/2]\mathrm{Ma}^2\right)}{\left[1+k\mathrm{Ma}^2\right]^2} \qquad (11.131)$$

Summary of Property Relations for Rayleigh Flow

$$\frac{p}{p_a} = \frac{1+k}{1+k\mathrm{Ma}^2} \qquad (11.123)$$

$$\frac{T}{T_a} = \left[\frac{(1+k)\mathrm{Ma}}{1+k\mathrm{Ma}^2}\right]^2 \qquad (11.128)$$

$$\frac{\rho_a}{\rho} = \frac{V}{V_a} = \mathrm{Ma}\left[\frac{(1+k)\mathrm{Ma}}{1+k\mathrm{Ma}^2}\right] \qquad (11.129)$$

$$\frac{p_0}{p_{0,a}} = \frac{p_0}{p}\frac{p}{p_a}\frac{p_a}{p_{0,a}} = \frac{(1+k)}{(1+k\mathrm{Ma}^2)}\left[\left(\frac{2}{k+1}\right)(1+[(k-1)/2]\mathrm{Ma}^2)\right]^{k/k-1} \qquad (11.133)$$

$$\frac{T_0}{T_{0,a}} = \frac{T_0}{T}\frac{T}{T_a}\frac{T_a}{T_{0,a}} = \frac{2(k+1)\mathrm{Ma}^2(1+[(k-1)/2]\mathrm{Ma}^2)}{[1+k\mathrm{Ma}^2]^2} \qquad (11.131)$$

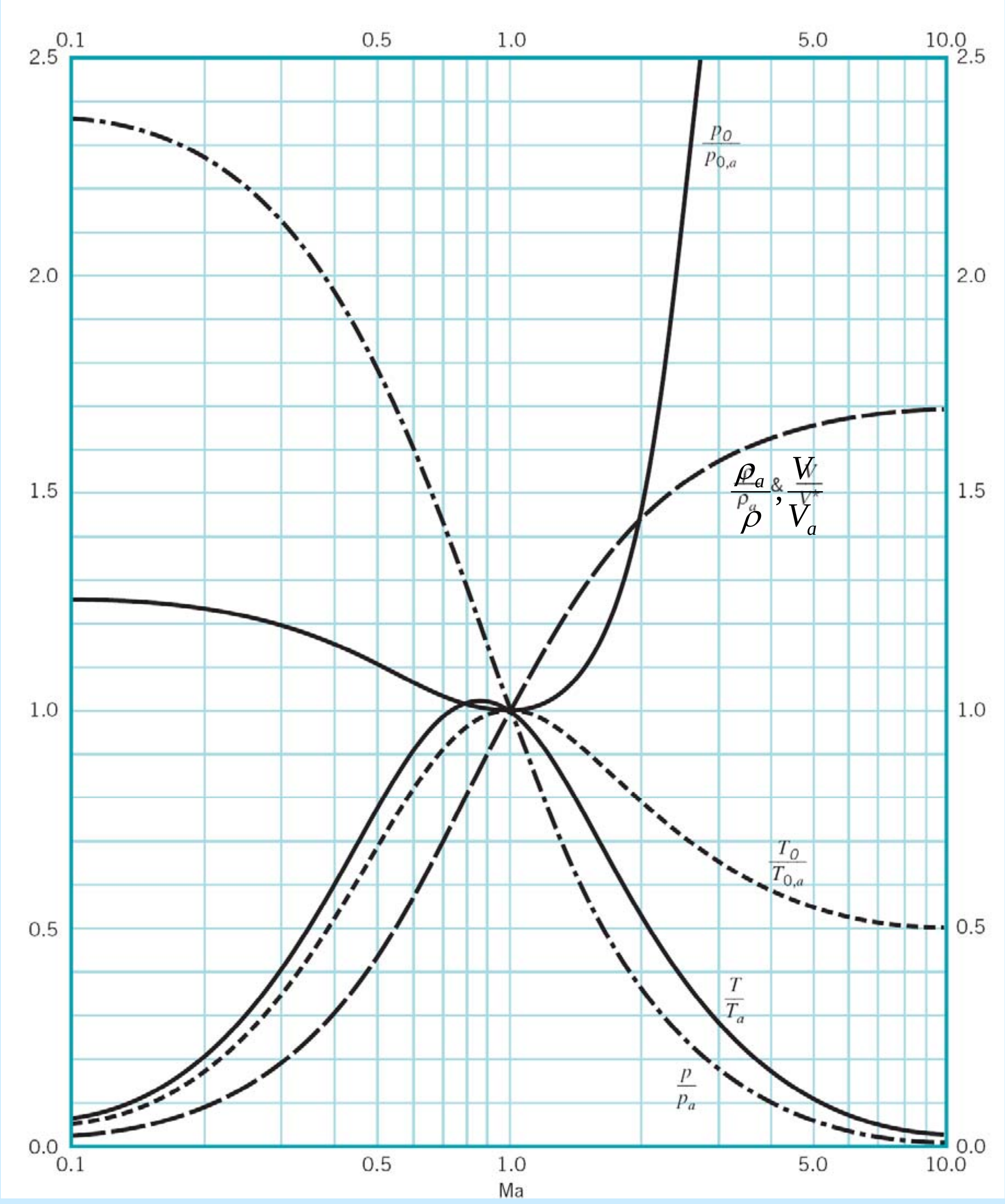


Figure D3 (p. 720)

Rayleigh flow of an idea gas with *k* = 1.4. (Graph provided by Dr. Bruce A. Reichert.)

Effect of Ma and Heating/Cooling for Rayleigh Flow (Example 11.16)

	Heating		Cooling	
	Subsonic	**Supersonic**	**Subsonic**	**Supersonic**
V	Increase	Decrease	Decrease	Increase
Ma	Increase	Decrease	Decrease	Increase
T	Increase for $0 \leq \text{Ma} \leq \sqrt{1/k}$ Decrease for $\sqrt{1/k} \leq \text{Ma} \leq 1$	Increase	Decrease for $0 \leq \text{Ma} \leq \sqrt{1/k}$ Increase for $\sqrt{1/k} \leq \text{Ma} \leq 1$	Decrease
T_0	Increase	Increase	Decrease	Decrease
p	Decrease	Increase	Increase	Decrease
p_0	Decrease	Decrease	Increase	Increase

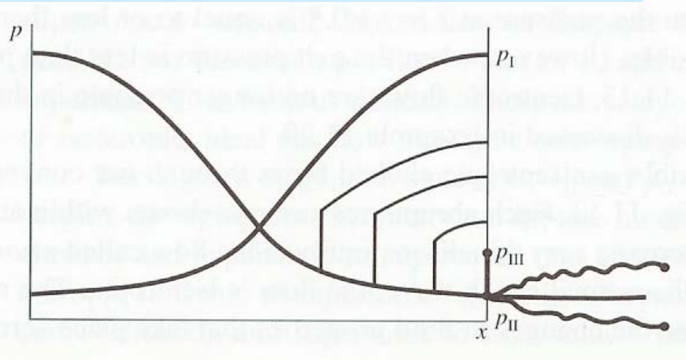
Note: p_0 is not constant, although frictionless

11.5.3 Normal Shock Waves

-Normal shock waves involves:

- deceleration from supersonic to subsonic
- a pressure rise
- an increase of entropy

(from Cengel and Cimbala, Fluid Mechanics, 2006)



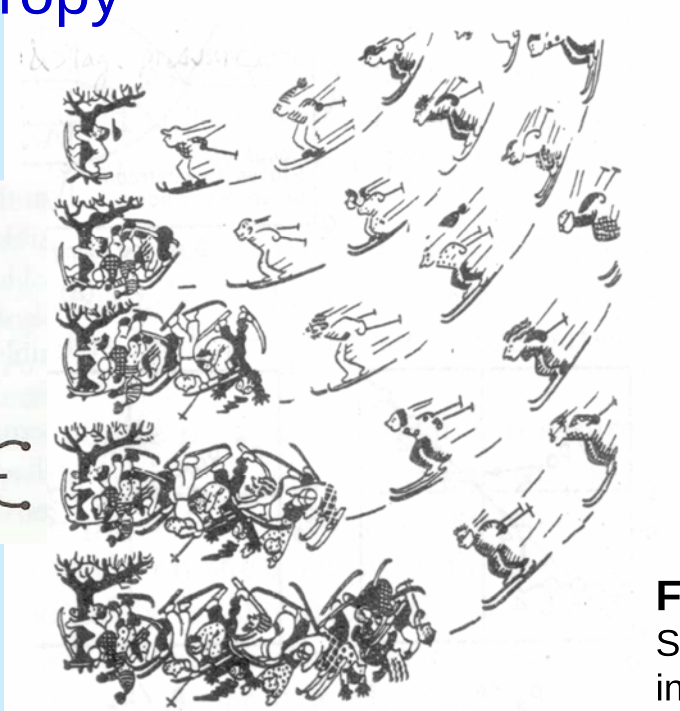


Fig. 5.20. Metaphorical shock propagating into a flowing medium which has been suddenly brought to rest. (Reprinted courtesy R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves*, Interscience Publishers, New York, copyright 1948.)

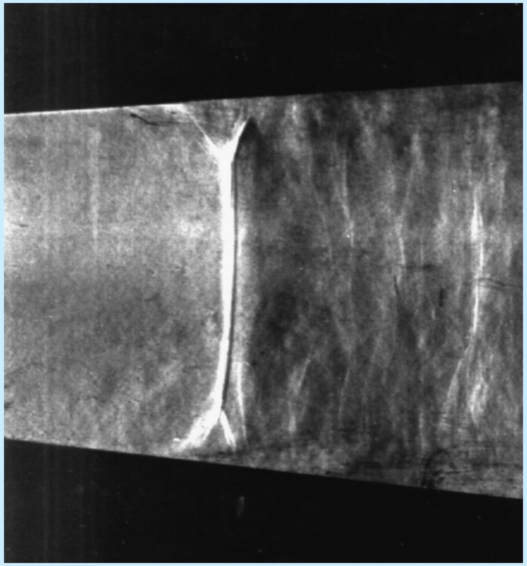


FIGURE 12–30
Schlieren image of a normal shock in a Laval nozzle. The Mach number in the nozzle just upstream (to the left) ofthe shock wave is about 1.3. Boundary layers distort the shape of the normal shock near the walls and lead to flow separation beneath the shock.

V11.6 Supersonic nozzle flow

V11.7 Blast waves

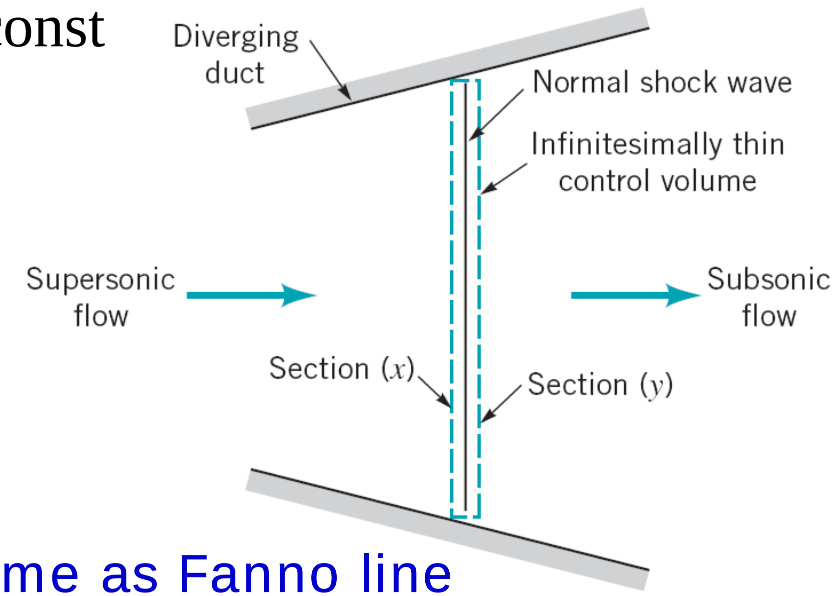
Derivation for Normal Shock Waves

- infinitesimal thin control volume surrounding the shock wave: friction and heat transfer negligible, *A*=*C*

—continuity: $\rho V = \text{const}$

—linear momentum（friction negligible）:—same as Rayleigh line

$$p + \rho V^2 = \text{const} \quad \text{or} \quad p + \frac{(\rho V)^2 RT}{p} = \text{const}$$



—energy: with $\Delta z = 0,\ \Delta q = 0$

$$h + \frac{V^2}{2} = h_0 = \text{const}$$

For an ideal gas,

$$T + \frac{(\rho V)^2 T^2}{2c_p\left(p^2 / R^2\right)} = T_0 = \text{const}$$

—same as Fanno line

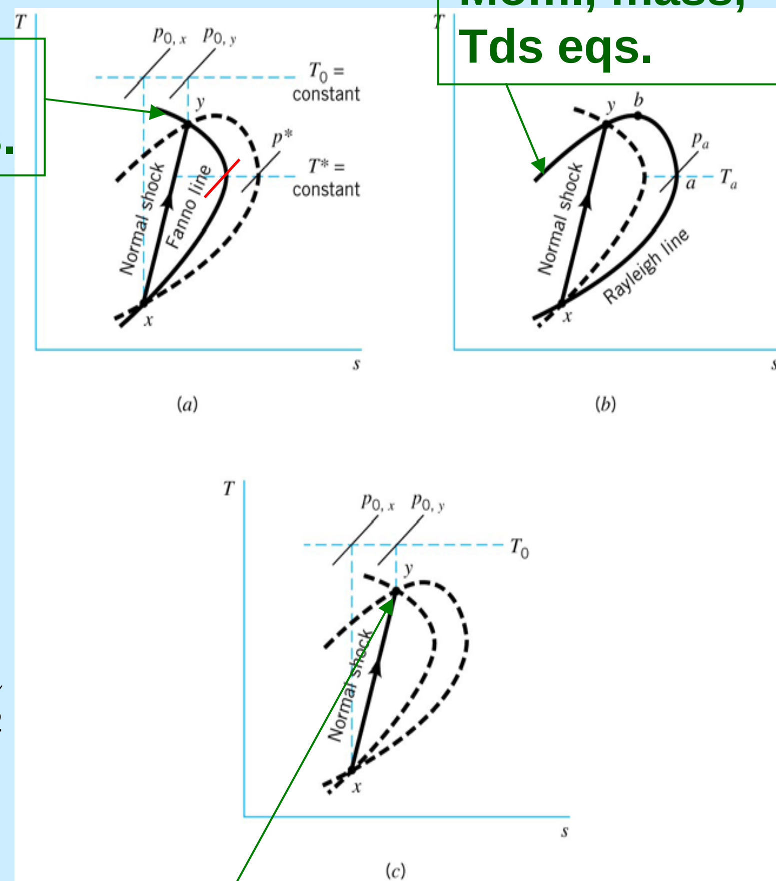
—*Tds* relationship: $s - s_1 = c_p \ell n \frac{T}{T_1} - R \ell n \frac{p}{p_1}$

Q: The irreversibility in shock wave is not from friction or heat transfer. What is it from?

Since shock wave flows have the same energy eq. for Fanno flows and same momentum eq. for Rayleigh flows, thus for a given ρV, gas (R, k), and conditions at the inlet of the normal shock (T_x, p_x, s_x), the conditions downstream of the shock (state y) will be on both a Fanno line and a Rayleigh line that pass through the inlet state (state x).

Total energy, mass, Tds eqs.

Mom., mass, Tds eqs.



Total energy, mom., mass, Tds eqs.

- For Rayleigh line,

$$\frac{p_y}{p_x} = \frac{p_y}{p_a}\frac{p_a}{p_x}$$

$$\because \frac{p_y}{p_a} = \frac{1+k}{1+k\mathrm{Ma}_y^2}, \quad \frac{p_x}{p_a} = \frac{1+k}{1+k\mathrm{Ma}_x^2}$$

$$\therefore \frac{p_y}{p_x} = \frac{1+k\mathrm{Ma}_x^2}{1+k\mathrm{Ma}_y^2} \qquad (11.140)$$

- For Fanno line

$$\frac{p_y}{p_x} = \frac{p_y}{p^*}\frac{p^*}{p_x} \quad \text{and} \quad \frac{p}{p^*} = \frac{1}{\mathrm{Ma}}\left[\frac{(k+1)/2}{1+[(k-1)/2]\mathrm{Ma}^2}\right]^{1/2}$$

$$\rightarrow \frac{p_y}{p_x} = \frac{\mathrm{Ma}_x}{\mathrm{Ma}_y}\left[\frac{1+[(k-1)/2]\mathrm{Ma}_x^2}{1+[(k-1)/2]\mathrm{Ma}_y^2}\right]^{1/2} \qquad (11.148)$$

Combining (11.140) and (11.148)

$$\rightarrow \frac{p_y}{p_x} = \left[\frac{1+[(k-1)/2]\mathrm{Ma}_x^2}{1+[(k-1)/2]\mathrm{Ma}_y^2}\right]^{1/2} \frac{\mathrm{Ma}_x}{\mathrm{Ma}_y} = \frac{1+k\mathrm{Ma}_x^2}{1+k\mathrm{Ma}_y^2}$$

$$\Rightarrow \mathrm{Ma}_y^2 = \frac{\mathrm{Ma}_x^2 + [2/(k-1)]}{[2k/(k-1)]\mathrm{Ma}_x^2 - 1} \qquad (11.149)$$

$$\text{(11.149) into (11.140)} \Rightarrow \frac{p_y}{p_x} = \frac{1+k\mathrm{Ma}_x^2}{1+k\mathrm{Ma}_y^2} = \frac{2k}{k+1}\mathrm{Ma}_x^2 - \frac{k-1}{k+1} \qquad (11.150)$$

- For Fanno line, Eq. 11.101

$$\frac{T_y}{T^*} = \frac{(k+1)/2}{1+[(k-1)/2]\mathrm{Ma}_y^2} \quad \text{and} \quad \frac{T_x}{T^*} = \frac{(k+1)/2}{1+[(k-1)/2]\mathrm{Ma}_x^2}$$

$$\rightarrow \frac{T_y}{T_x} = \frac{T_y}{T^*}\frac{T^*}{T_x} = \frac{1+[(k-1)/2]\mathrm{Ma}_x^2}{1+[(k-1)/2]\mathrm{Ma}_y^2} \qquad (11.144)$$

$$\text{(11.149) into (11.144)} \Rightarrow \frac{T_y}{T_x} = \frac{[1+[(k-1)/2]\mathrm{Ma}_x^2]\{[2k/(k-1)]\mathrm{Ma}_x^2 - 1\}}{\{(k+1)^2/[2(k-1)]\}\mathrm{Ma}_x^2} \qquad (11.151)$$

Summary of Property Relations across Shock Wave

If Ma_x is known, property ratios across the shock can be known:

$$Ma_y^2 = \frac{Ma_x^2 + [2/(k-1)]}{[2k/(k-1)]Ma_x^2 - 1} \quad (11.149)$$

$$\frac{p_y}{p_x} = \frac{2k}{k+1} Ma_x^2 - \frac{k-1}{k+1} \quad (11.150)$$

$$\frac{T_y}{T_x} = \frac{[1+[(k-1)/2]Ma_x^2]\{[2k/(k-1)]Ma_x^2 - 1\}}{\{(k+1)^2/[2(k-1)]\}Ma_x^2} \quad (11.151)$$

$$\frac{\rho_y}{\rho_x} = \frac{V_x}{V_y} = \frac{(k+1)Ma_x^2}{(k-1)Ma_x^2 + 2} \quad (11.154)$$

$$\frac{p_{0,y}}{p_{0,x}} = \frac{[\frac{k+1}{2}Ma_x^2]^{\frac{k}{k-1}}[1+\frac{k-1}{2}Ma_x^2]^{\frac{k}{k-1}}}{[\frac{2k}{k+1}Ma_x^2 - \frac{k-1}{k+1}]^{\frac{1}{k-1}}} \quad (11.156)$$

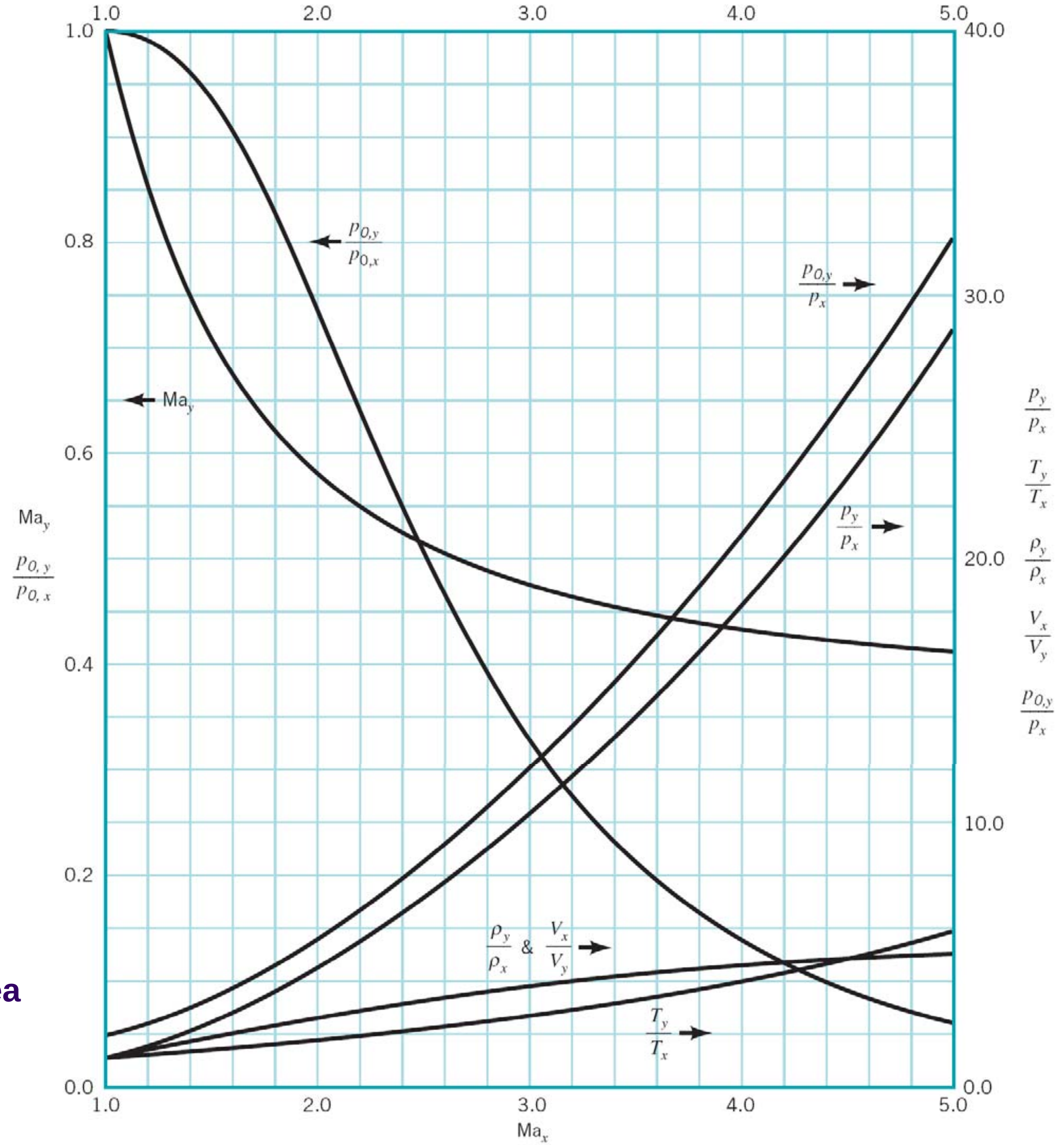


Figure D4 (p. 721)

Normal shock flow of an idea gas with *k* = 1.4. (Graph provided by Dr. Bruce A. Reichert.)

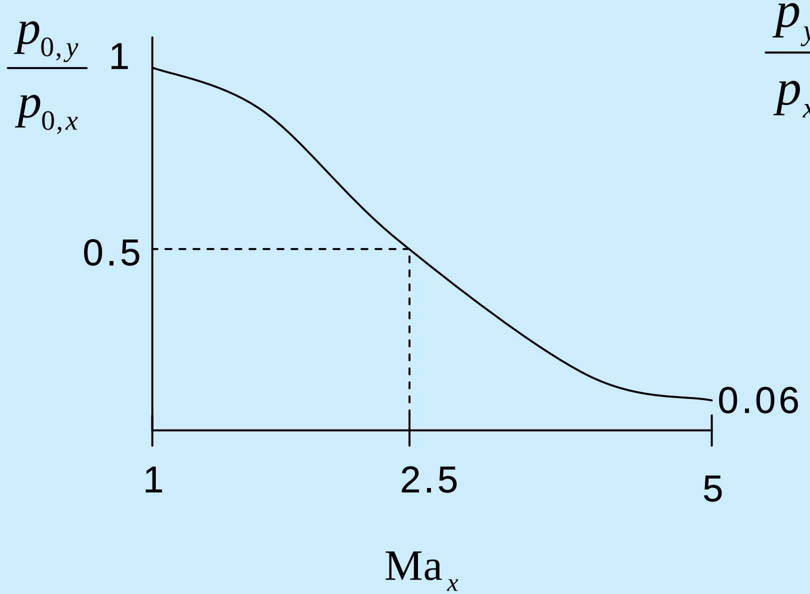
TABLE 11.3

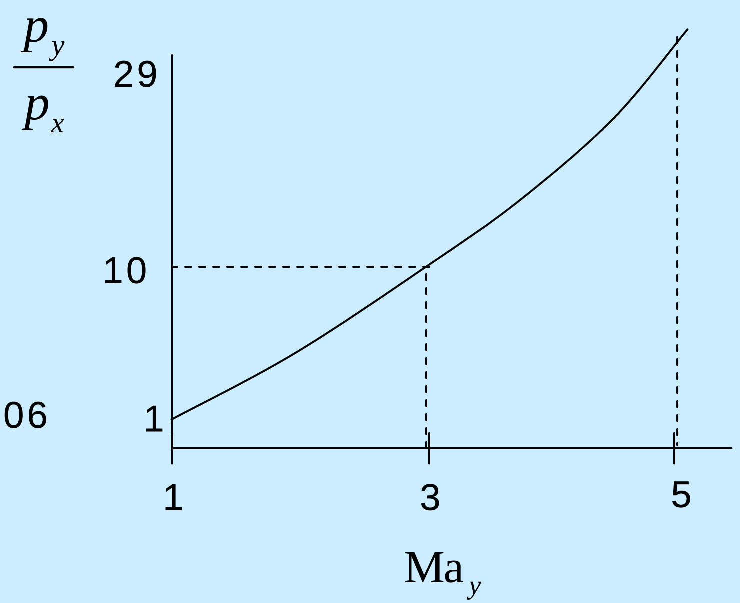
Summary of Normal Shock Wave Characteristics

Variable	Change Across Normal Shock Wave
Mach number	Decrease
Static pressure	Increase
Stagnation pressure	Decrease
Static temperature	Increase
Stagnation temperature	Constant
Density	Increase
Velocity	Decrease

Example 11.17 Stagnation pressure drop across a normal shock

$$\text{For } \mathrm{Ma}_x \uparrow \Rightarrow \quad \frac{p_y}{p_x}\uparrow \quad \text{and} \quad \frac{p_{0,y}}{p_{0,x}}\downarrow$$





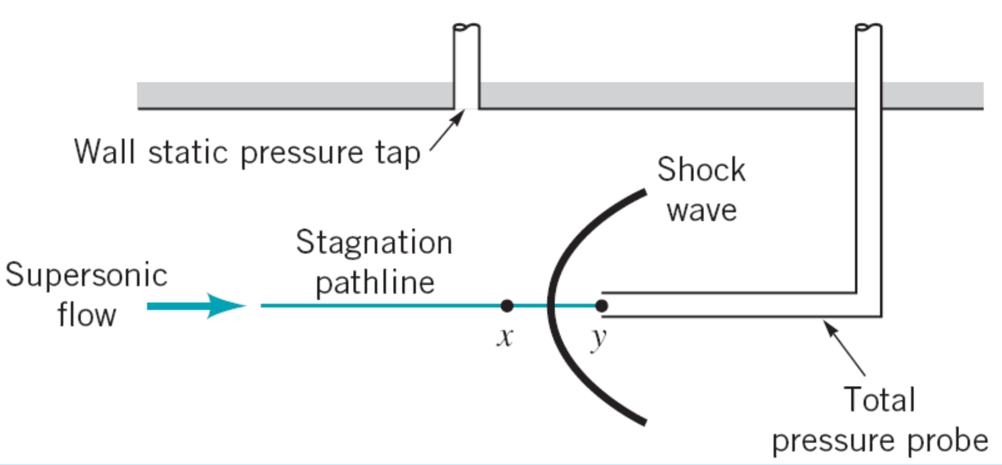
Across a normal shock, adverse pressure gradient occurs which can cause flow separation. Therefore, shock-boundary layer interactions are of great concern to designers of high speed flow device.

Example 11.18 Supersonic flow pitot tube

Given: $p_{0,y} = 4114\,\text{kPa},\ T_0 = 555\text{K},\ p_x = 82\,\text{kPa}$

$$\frac{p_{0,y}}{p_x} = \frac{p_{0,y}}{p_{0,x}}\frac{p_{0,x}}{p_x} = \frac{\left[\frac{k+1}{2}\text{Ma}_x^2\right]^{\frac{k}{k-1}}}{\left[\frac{2k}{k+1}\text{Ma}_x^2 - \frac{k-1}{k+1}\right]^{\frac{1}{k-1}}} \Leftarrow \text{Rayleigh Pitot tube formula}$$

$$= \frac{414\text{kPa}}{82\text{kPa}} = 5$$



$$\text{Fig. D1} \rightarrow \text{Ma}_x = 1.9$$

$$V_x = \text{Ma}_x c_x = \text{Ma}_x\sqrt{kRT_x}$$

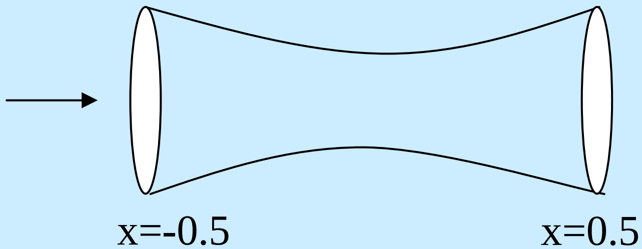
$$T_{0,x} = T_{0,y},\quad \frac{T_x}{T_{0,x}} = 0.59 \Rightarrow T_x = 327\text{K}$$

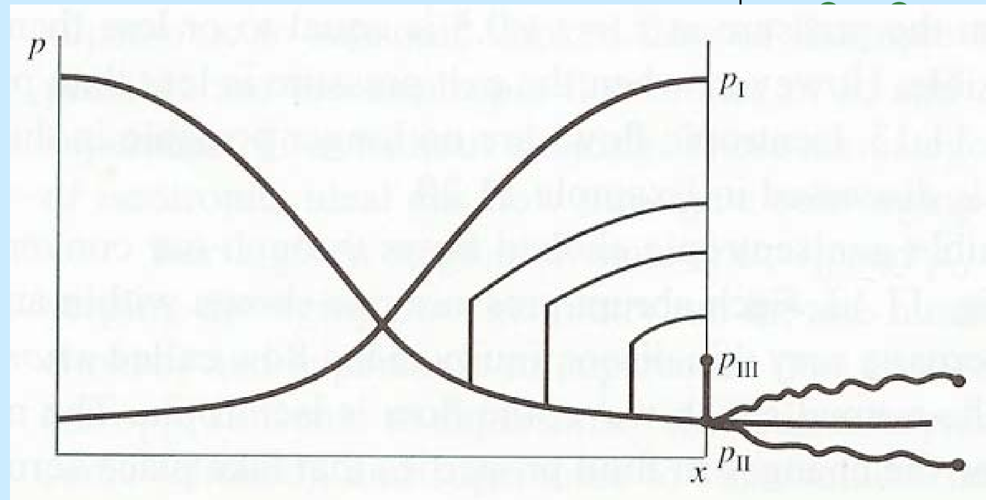
$$\Rightarrow V_x = 678\ \text{m/s}$$

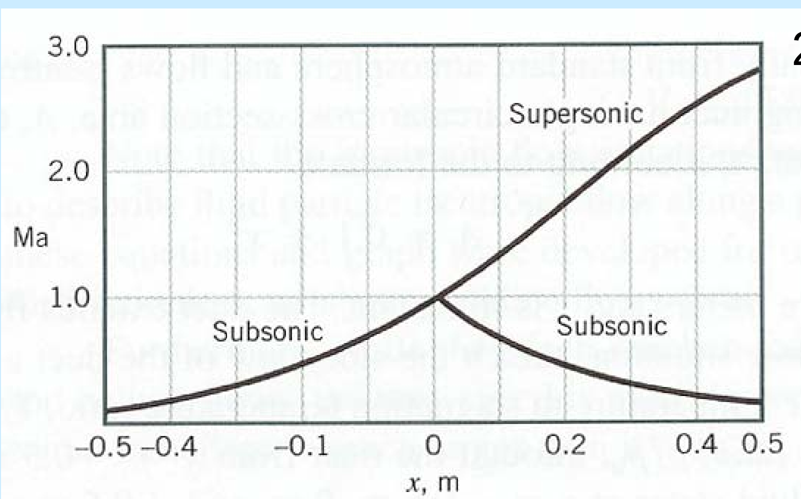
Note: Incompressible calculation for pitot tube would give the wrong result.

Example 11.19 Normal shock in a converging-diverging duct

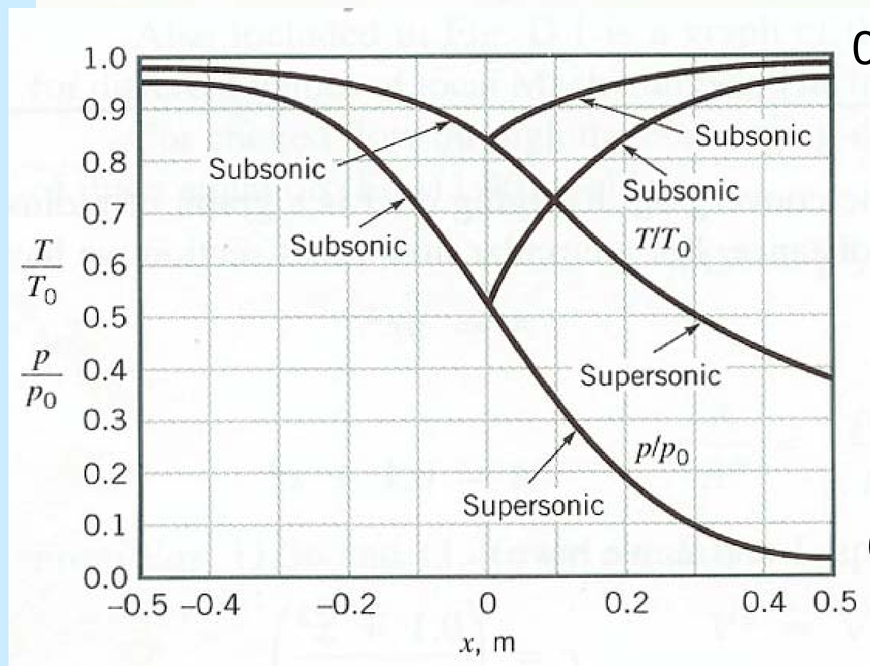
(a) $\dfrac{p_{III}}{p_0} = ?$ for normal shock at exit, (b) $\dfrac{p}{p_0} = ?$ for shock at $x = 0.3$ m







2.8



0.98

0.04

-shock at x=0.5m

From Ex 11.8: $\text{Ma}_x = 2.8$ and $\dfrac{p_x}{p_{0,x}} = 0.04$ at $x = 0.5\text{m}$

Fig. D4 $\rightarrow \text{Ma}_x = 2.8$ (at exit): $\dfrac{p_y}{p_x} = 9, \ \dfrac{p_{0,y}}{p_{0,x}} = 0.38$

$$\frac{p_y}{p_{0,x}} = \frac{p_y}{p_x}\frac{p_x}{p_{0,x}} = 9 \times 0.04 = 0.36 = \frac{p_{III}}{p_{0,x}}$$

$$\frac{p_{0,y}}{p_{0,x}} = 0.38 \ \text{– considerable energy loss}$$

-shock at x=0.3m

From Ex 11.8: $\text{Ma}_x = 2.14, \ \dfrac{p_x}{p_{0,x}} = 0.1$

Fig. D4 with $\text{Ma}_x = 2.14 \rightarrow$ across the shock

$$\frac{p_y}{p_x} = 5.2, \ \text{Ma}_y = 0.56, \ \frac{p_{0,y}}{p_{0,x}} = 0.66$$

(Conditions downstream the shock at x=0.3m)

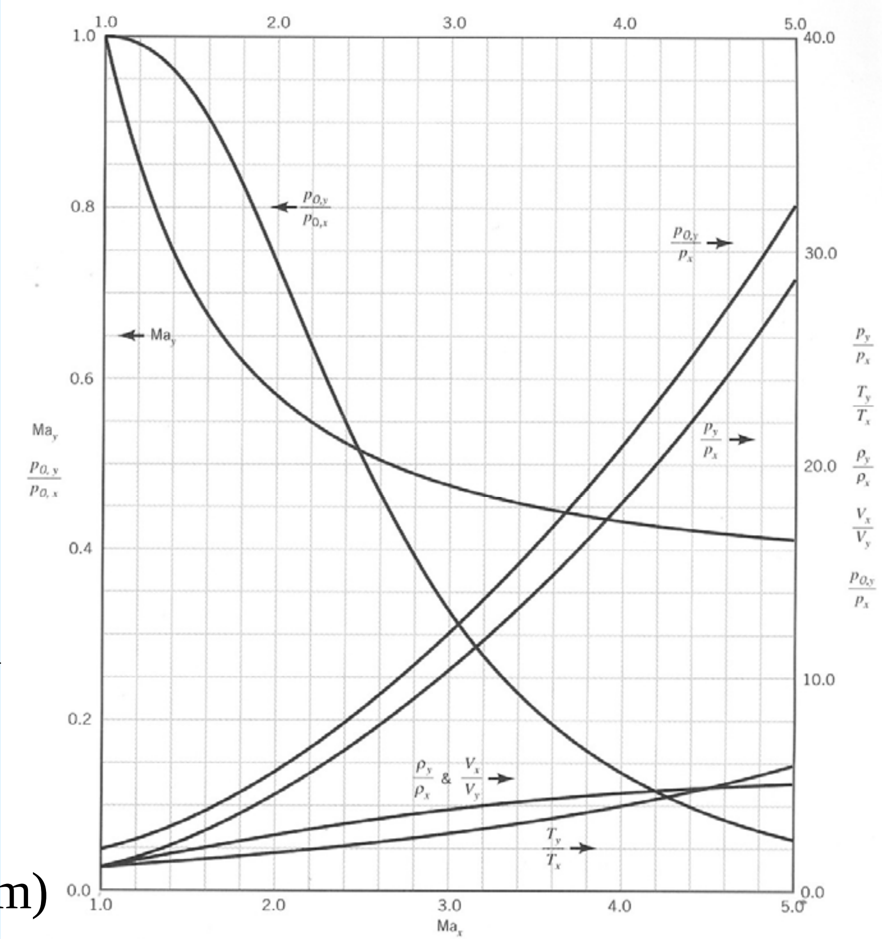


FIGURE D.4 Normal shock flow of an ideal gas with $k = 1.4$. (Graph provided by Professor Bruce A. Reichert of Kansas State University.)

Note: for isentropic A*=0.1 (a minimum)

Consider the isentropic decelerating flow downstream the shock

Also, Fig. D4 $\rightarrow \ \dfrac{A_y}{A^*} = 1.24$ (Here, A^* is used as dummy)

$$\frac{A_2}{A_y} = \frac{0.1 + (0.5)^2}{0.1 + (0.3)^2} = 1.842$$

$$\frac{A_2}{A^*} = \frac{A_y}{A^*}\frac{A_2}{A_y} = 1.24 \times 1.842 = 2.28$$

$$\rightarrow A^* = \frac{A_2}{2.28} = 0.15$$

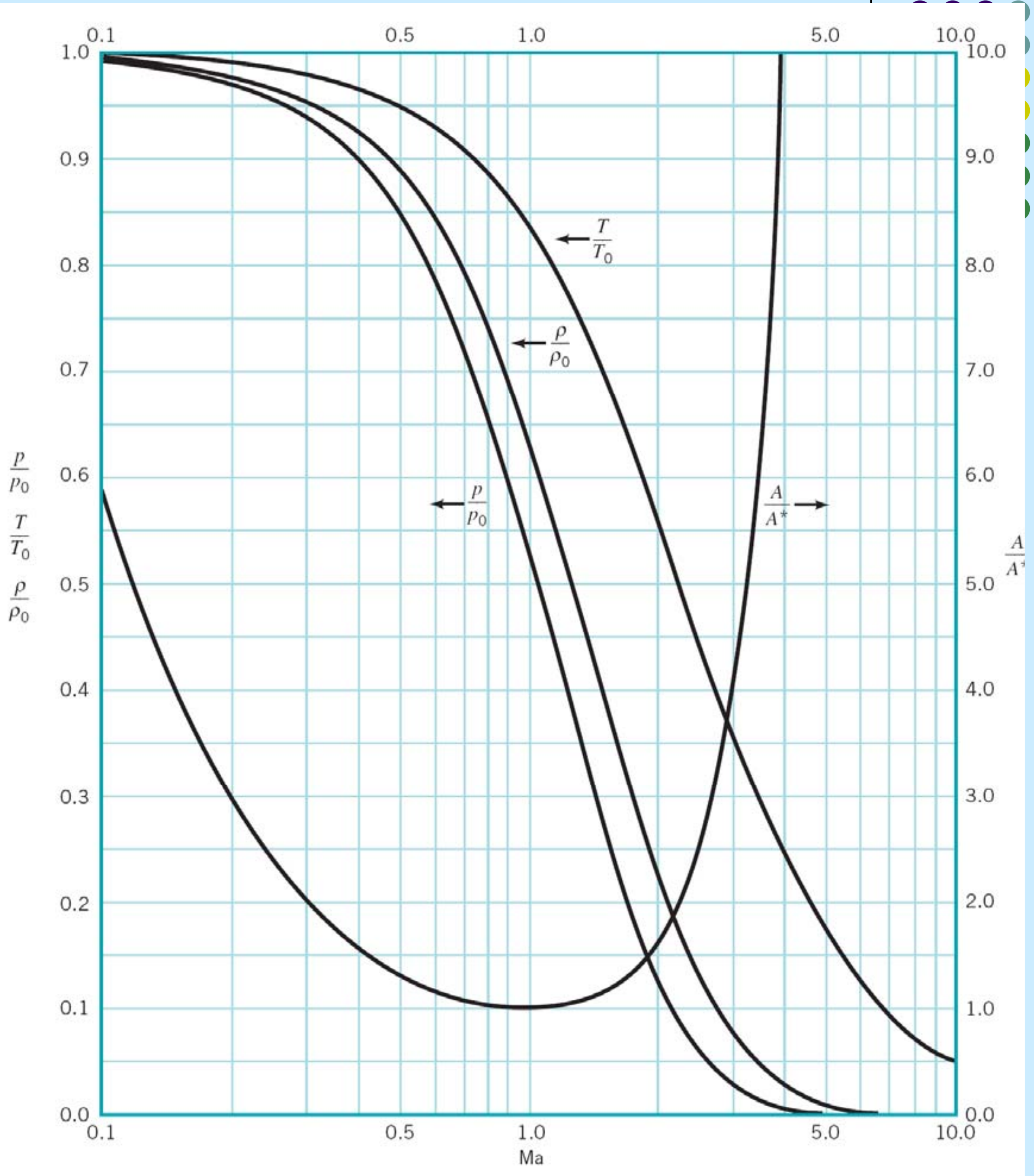
$\dfrac{A_2}{A^*} = 2.28,$ Fig. D1 (isentropic) $\rightarrow \mathrm{Ma}_2 = 0.26, \quad \dfrac{p_2}{p_{0,y}} = 0.95$

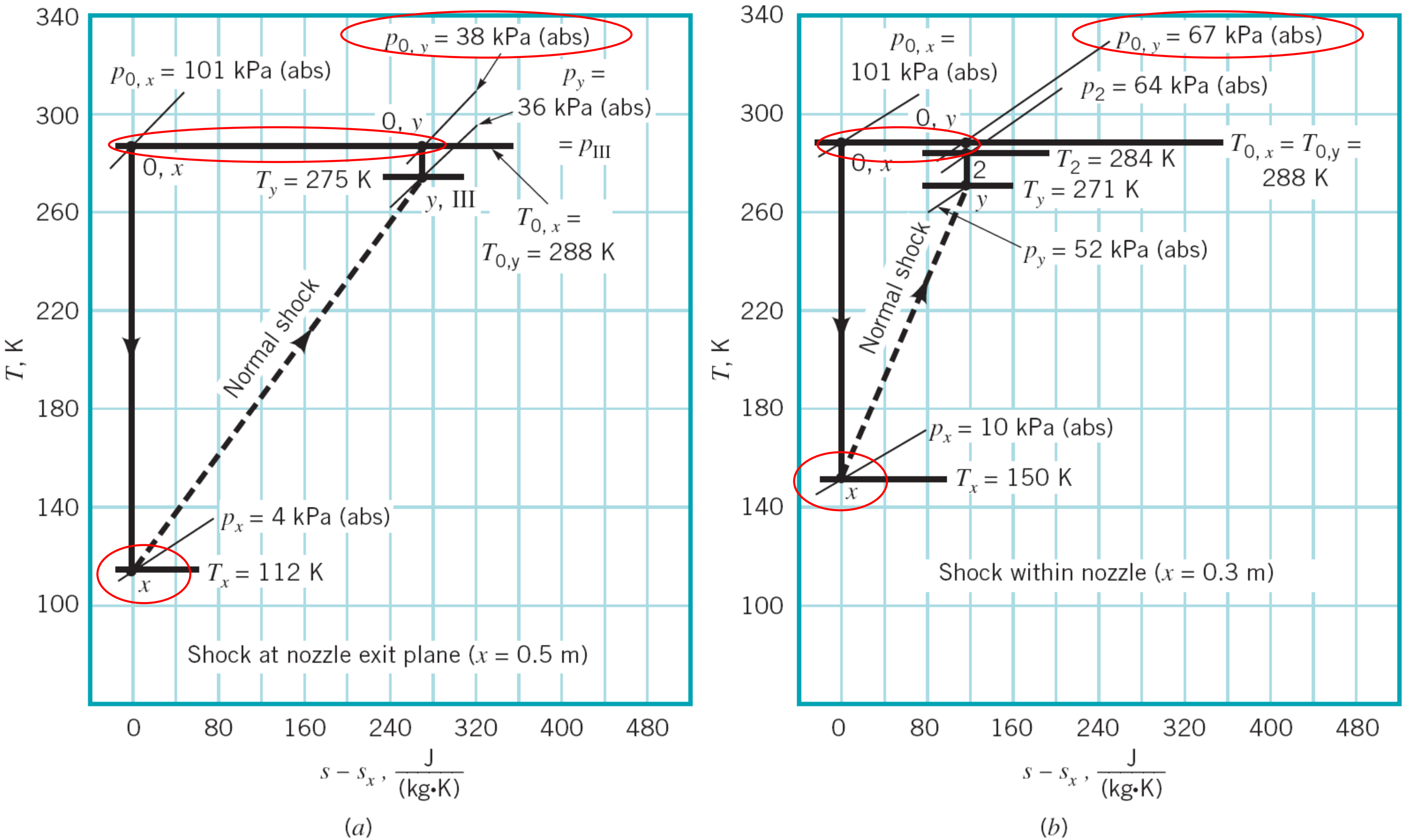
$$\rightarrow \frac{p_2}{p_{0,x}} = \frac{p_2}{p_{0,y}}\frac{p_{0,y}}{p_{0,x}} = 0.95 \times 0.66 = 0.63$$

Note: $\dfrac{p_{0,y}}{p_{0,x}} = 0.66$, shock at $x = 0.3\text{m} > \dfrac{p_{0,y}}{p_{0,x}} = 0.36$, shock at exit $x = 0.5\text{m}$

Figure D1 (p. 718)

Isentropic flow of an ideal gas with *k* = 1.4. (Graph provided by Dr. Bruce A. Reichert.)



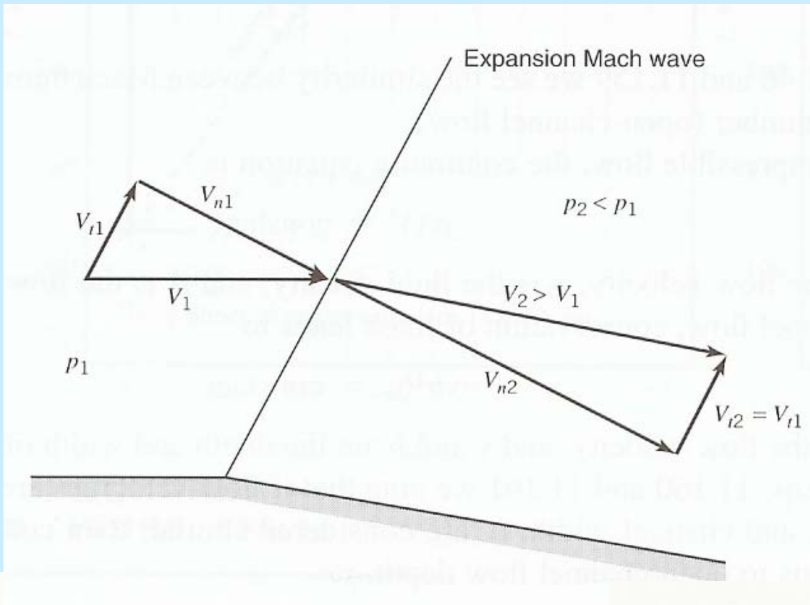


340
300
260
220
180
140
100
T, K
p0, x = 101 kPa (abs)
p0, y = 38 kPa (abs)
0, y
py =
36 kPa (abs)
= pIII
0, x
Ty = 275 K
y, III
T0, x =
T0,y = 288 K
Normal shock
px = 4 kPa (abs)
Tx = 112 K
x
Shock at nozzle exit plane (x = 0.5 m)
0 80 160 240 320 400 480
s − sx , J/(kg·K)
(a)
p0, x =
101 kPa (abs)
p0, y = 67 kPa (abs)
p2 = 64 kPa (abs)
0, y
T0, x = T0,y =
288 K
0, x
2
T2 = 284 K
Ty = 271 K
y
py = 52 kPa (abs)
Normal shock
px = 10 kPa (abs)
Tx = 150 K
x
Shock within nozzle (x = 0.3 m)
s − sx , J/(kg·K)
(b)

11.7 Two-Dimensional Compressible Flow

Supersonic

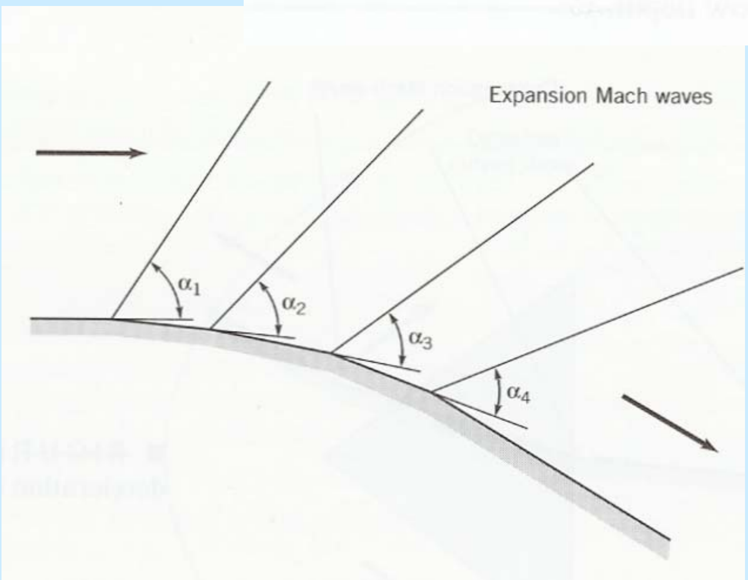
- Flow acceleration across a Mach wave

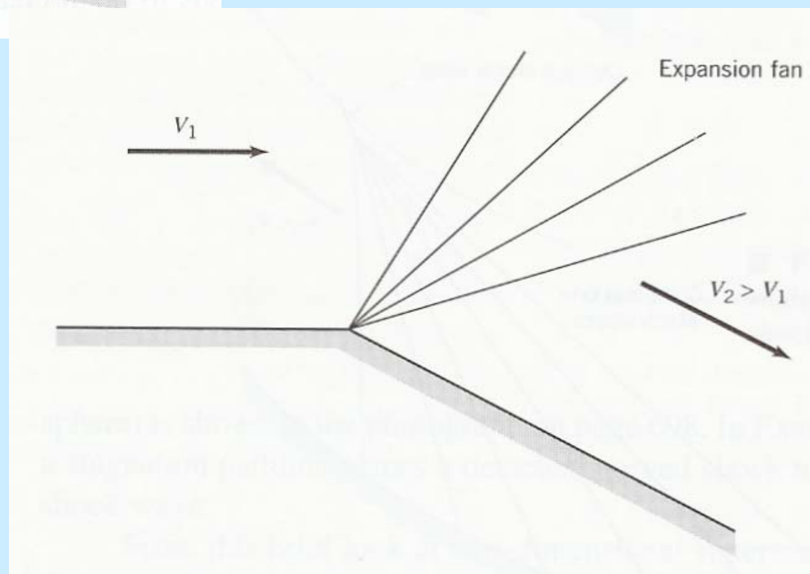


$$V_{t_1} = V_{t_2}$$

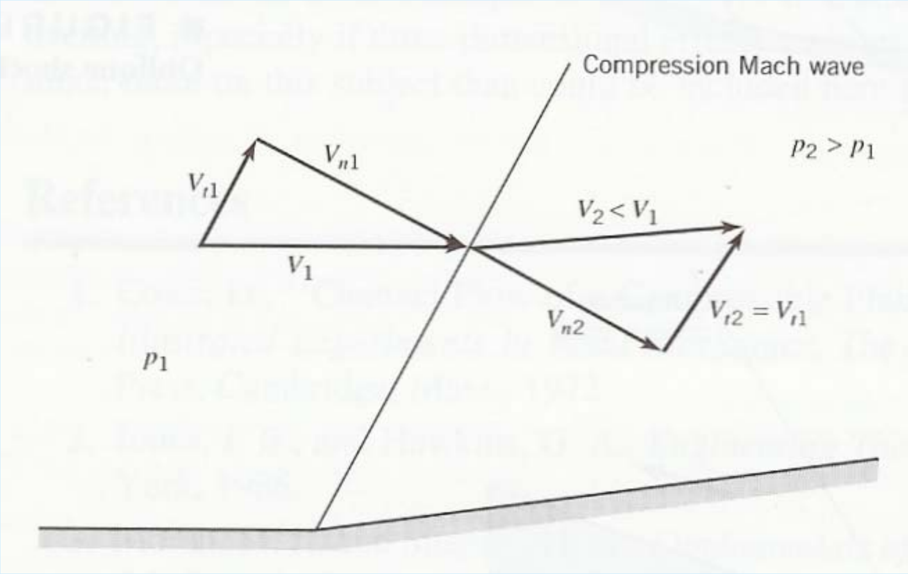
$$V_{n_2} > V_{n_1}$$

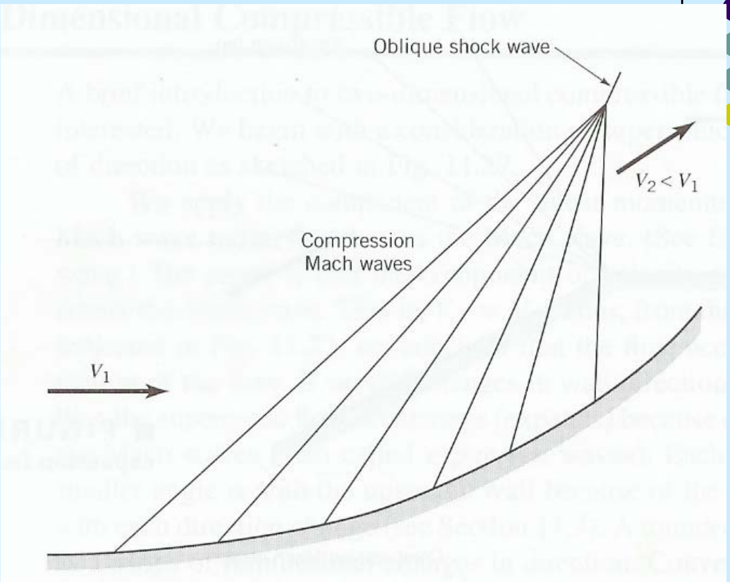
$$\therefore V_2 > V_1$$



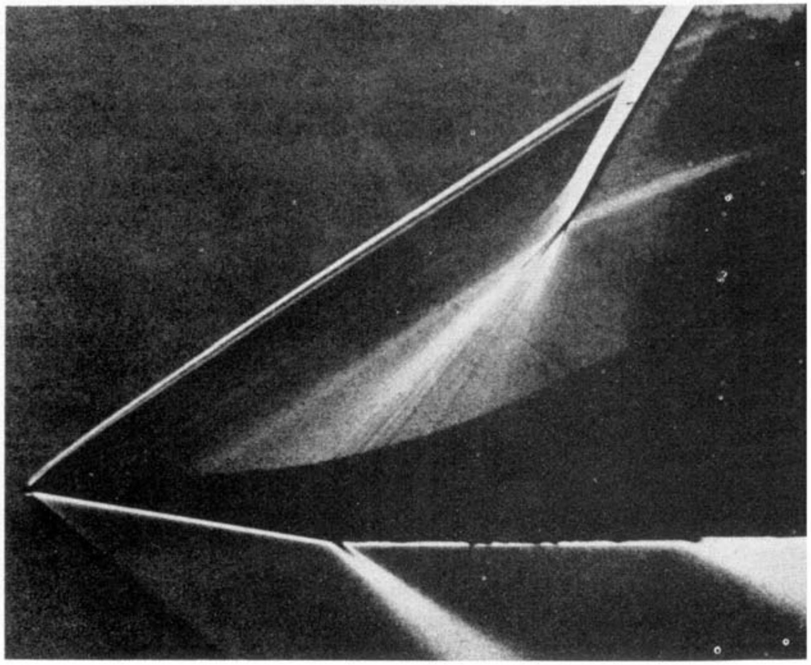


- Supersonic flows decelerate across compression Mach wave





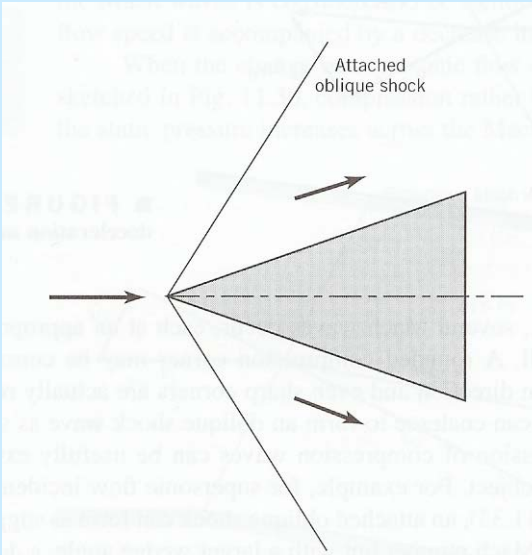
227. Steady formation of an oblique shock wave. A cylindrical concave surface in a supersonic wind tunnel at Mach number 1.96 produces a converging fan of compression waves, which are made visible by schlieren photography with the knife edge parallel to the free stream. They focus roughly as a centered compression, forming a strong oblique shock wave that turns the stream through 22.5 degrees. The surface extends upstream as a flat plate at zero incidence so that the weak shock wave from the slightly blunt leading edge will not obscure the view. The surface is roughened to make the boundary layer turbulent, so that it will not separate. *Johannesen 1952*



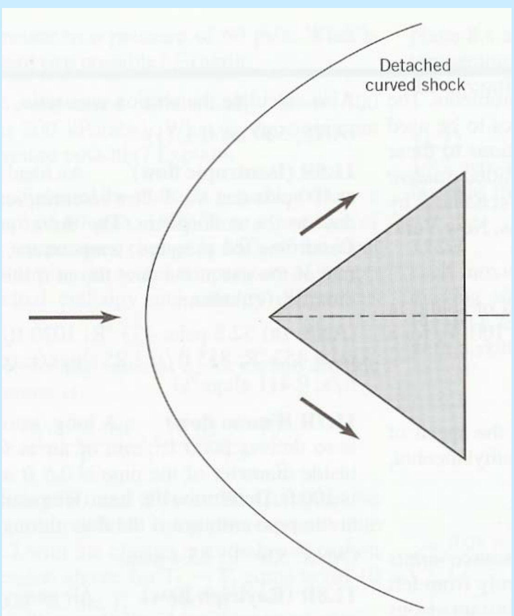
from M. Van Dyke, An Album of Fluid Motion

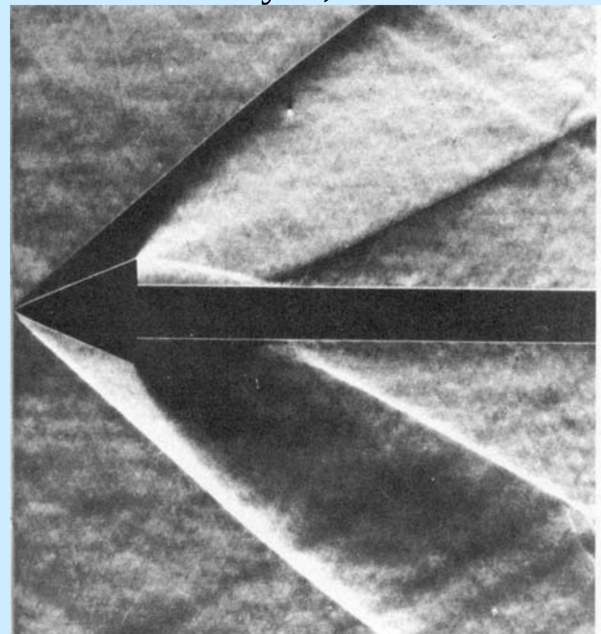
Small wedge angle-attached shock

from M. Van Dyke, An Album of Fluid Motion

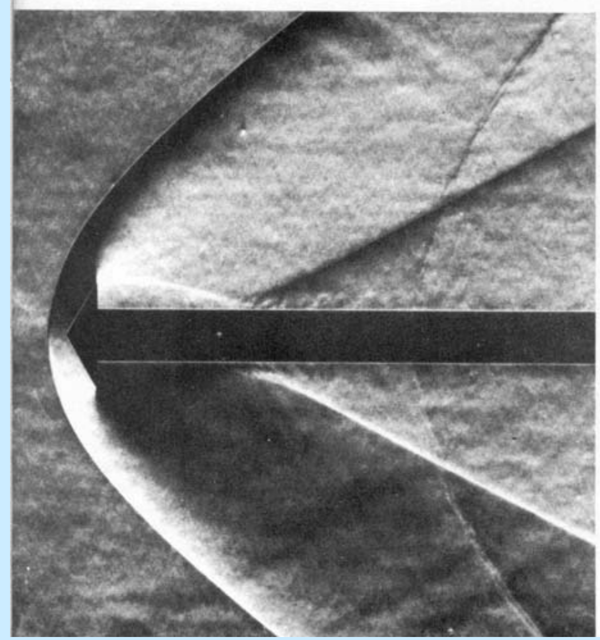


Large wedge angle-detached shock





257. Attached bow wave on a 22.5° cone at M=1.96. Here the flow is supersonic everywhere between the bow shock wave and the surface of the cone. Hence the field is conical and the wave straight until it is intersected by the expansion wave from the base of the cone. The schlieren knife edge is horizontal, making the image antisymmetric from top to bottom. *Photograph by A. W. Sharp, courtesy of N. Johannesen*



258. Detached bow wave on a 60° cone at M=1.96. At this Mach number in air the bow shock wave cannot remain attached to the vertex of a cone above a semi-vertex angle of 40°. It detaches, forming an embedded subsonic zone extending back to the cone surface. The two crinkled curves downstream are the intersections of the bow wave with the turbulent boundary layers on the wind-tunnel windows. *Photograph by A. W. Sharp, courtesy of N. Johannesen*

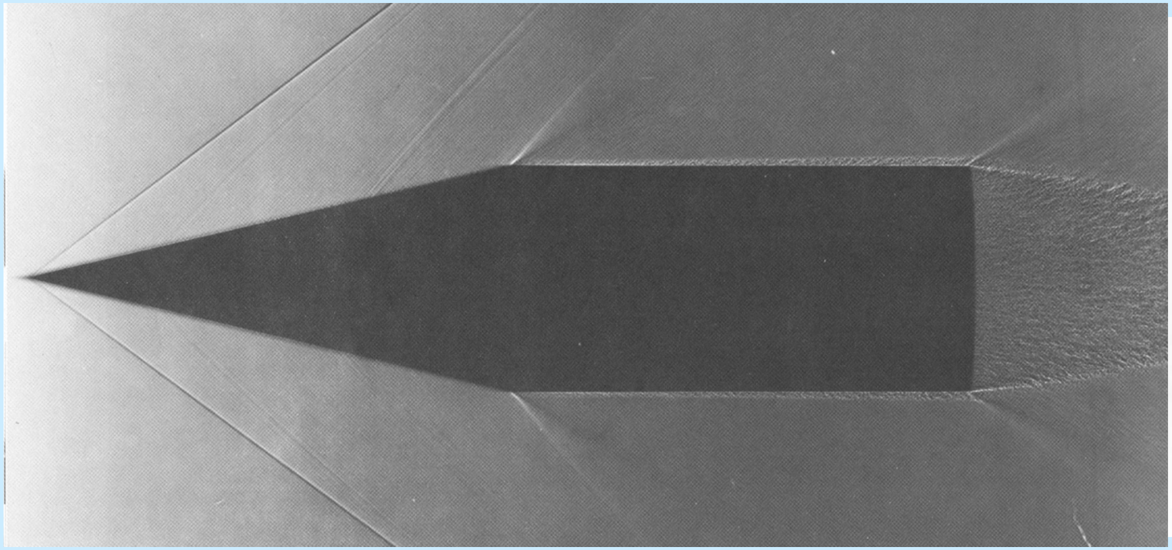


FIGURE 12–46
A cone-cylinder of 12.5° half-angle in a Mach number 1.84 flow. The boundary layer becomes turbulent shortly downstream of the nose, generating Mach waves that are visible in this shadowgraph. Expansion waves are seen at the corners and at the trailing edge of the cone.
Photo by A. C. Charters, as found in Van Dyke (1982).

(from Cengel and Cimbala, Fluid Mechanics, 2006)

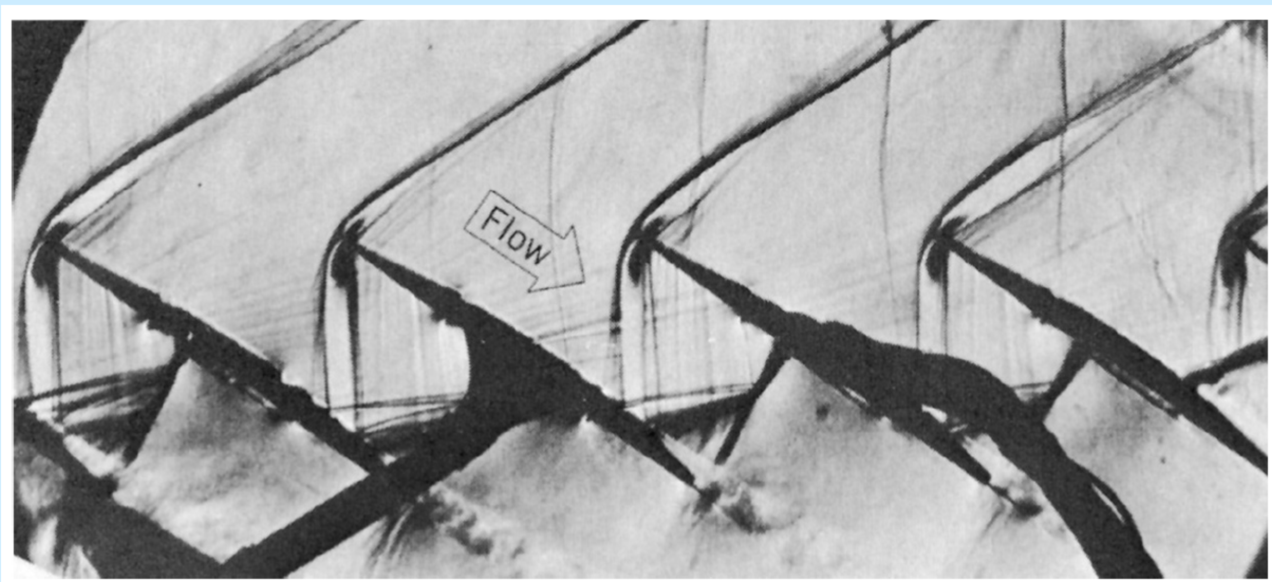
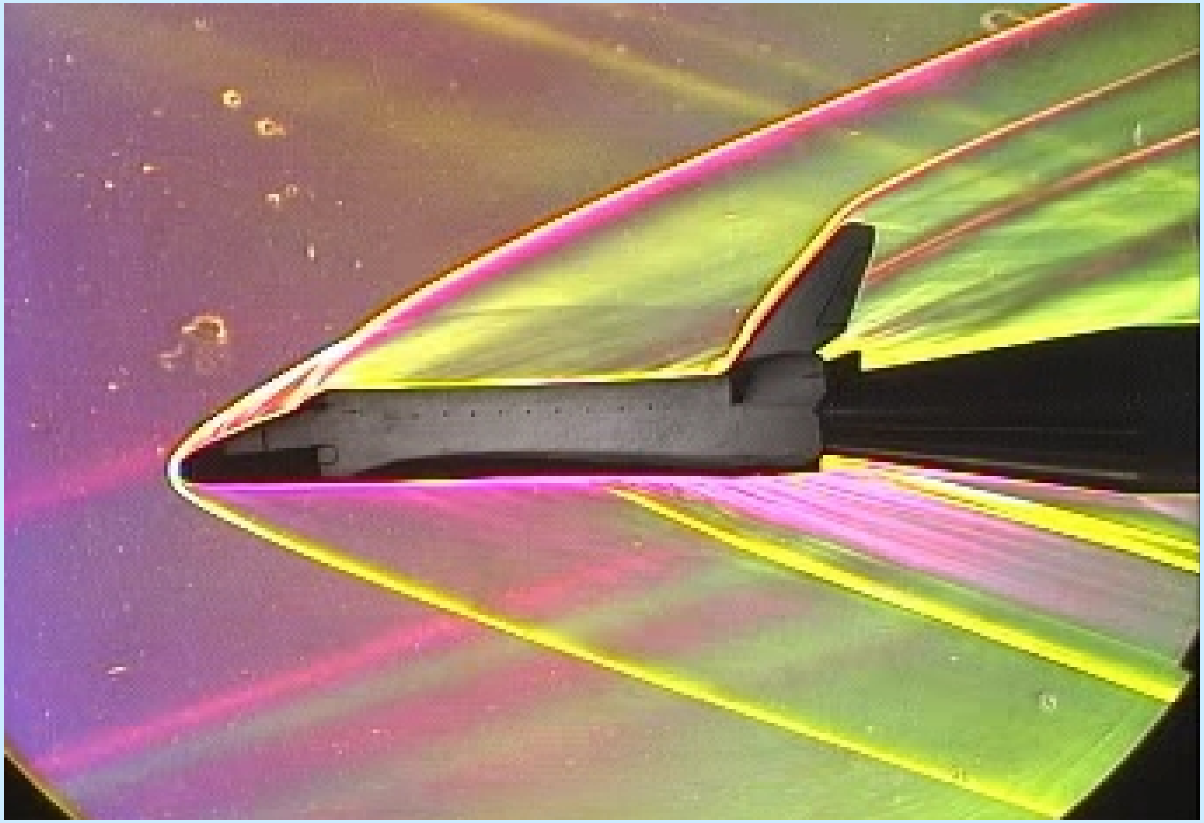


Figure 11.4 (p. 591)

The schlieren visualization of flow (supersonic to subsonic) through a row of compressor airfoils. (Photograph provided by Dr. Hans Starken, Germany.)

3-D shock wave around a model space shuttle



from Gas Dynamics Lab, The Penn. State University, 2004